On Moishezon Spaces that are Compactifications of
Reductive Groups

Jürgen Hausen

Konstanzer Schriften in Mathematik und Informatik
Nr. 14, September 1996
ISSN 1430–3558
On Moishezon Spaces that are
Compactifications of Reductive Groups

Jürgen Hausen\(^1\)

14 / 1996

Fakultät für Mathematik und Informatik
Universität Konstanz

1 Introduction

A compact complex space \(X\) is called a Moishezon space if each irreducible component \(X_i\)
of \(X\) has \(\dim \mathbb{C} X_i\) algebraically independent global meromorphic functions. A point \(x\) of a
Moishezon space \(X\) is schematic if there exist meromorphic functions on \(X\) that provide
local coordinates at \(x\).

In general the set of non schematic points of a normal Moishezon space \(X\) is analytic of
codimension at least 2, and \(X\) is algebraic if and only if every \(x \in X\) is schematic. So it is
natural to look for criteria for a point \(x \in X\) to be schematic.

In the present article we consider normal Moishezon spaces \(X\) that are equivariant comp-
actifications of a reductive complex Lie group \(G\) in the following sense: \(G\) acts algebraically
on \(X\) (see section 2) and there exists a point \(x_0 \in X\) with open dense orbit \(Gx_0\) and trivial
isotropy group \(G_{x_0}\).

D. Luna proved (see [Lu], Proposition): If a smooth Moishezon space \(X\) is an equivariant
compactification of \(G\) then every fixed point of \(G\) is schematic. The purpose of this note
is to generalize Luna’s result to a wider class of points. As a corollary of our result (see
section 4) we obtain:

Suppose that the normal Moishezon space \(X\) is an equivariant compactification of the
reductive Lie group \(G\) and let \(p \in X\). If the isotropy group \(G_p\) of \(p\) is reductive then \(p\) is
schematic.

The author would like to thank G. Barthel, A. Białynicki-Birula and L. Kaup for both,
helpful discussions and the suggestion to work on this subject.

\(^1\)email: Juergen.Hausen@uni-konstanz.de
2 Algebraic Group Actions on Moishezon Spaces

The purpose of this section is to define the concept of an algebraic action of an algebraic group on a Moishezon space. For the convenience of the reader we first recall some basic properties of Moishezon spaces in an elementary setting. A more advanced treatment of these facts is given in [Kn].

Let $X$ be a Moishezon space of complex dimension $n$ and let $x_0$ be a schematic point of $X$. Then, by definition, there are meromorphic functions $f_1, \ldots, f_n$ on $X$ that define local coordinates near $x_0$. Consider the meromorphic map

$$F : X \to \mathbb{P}_n, \quad x \mapsto [1, f_1(x), \ldots, f_n(x)]$$

(for an introduction to meromorphic maps see for example [Do], section 2, or [Ue], section I.2). As in [Do], proof of Proposition 5, we see that the closure of the graph $\Gamma(F) \subset X \times \mathbb{P}_n$ is a projective variety. Let

$$\text{pr}_X : \overline{\Gamma(F)} \to X, \quad (x, y) \mapsto x.$$ 

Denote by $U \subset X$ the maximal (open) set such that $x_0 \in U$ and the restriction of $\text{pr}_X$ to $\text{pr}_X^{-1}(U)$ is an isomorphism onto its image. Then we obtain (see [Do], Proposition 2):

2.1 Remark. Every $x \in U$ is schematic and $X \setminus U$ is an analytic set in $X$. If $X$ is normal then $X \setminus U$ is of codimension at least two. □

By Chow’s Theorem we obtain that $\text{pr}_X^{-1}(U)$ is a Zariski open subset of the projective variety $\overline{\Gamma(F)}$. In other words $U$ is biholomorphically equivalent to an algebraic variety. Moreover, if $x'$ is a second schematic point with local coordinates $f'_1, \ldots, f'_n$ then for the associated algebraic chart $\text{pr}'_X$ we obtain that the transition map $\text{pr}_X^{-1} \circ \text{pr}'_X$ is algebraic. So we conclude

2.2 Proposition. The set $X_{\text{sch}}$ of schematic points of $X$ is an algebraic variety. Moreover $A := X \setminus X_{\text{sch}}$ is an analytic set in $X$ and if $X$ is normal then $A$ is of codimension at least two. □

Now suppose we have a holomorphic action $G \times X \to X$ of an algebraic group $G$ on $X$. Then the set of schematic points of $X$ is invariant under this action. We call the action of $G$ on $X$ algebraic, if the restricted action $G \times X_{\text{sch}} \to X_{\text{sch}}$ is algebraic.
3 Meromorphic Continuation of $K$-finite Functions

Let $X$ be a reduced complex space with countable topology and let $G$ be a reductive complex Lie group acting holomorphically on $X$. We fix a maximal compact subgroup $K$ of $G$ (consequently $G$ is isomorphic to $K^\mathbb{C}$, the complexification of $K$).

For every $K$-invariant open set $U \subset X$ one defines a representation of $K$ on the complex vector space $\mathcal{O}(U)$ of holomorphic functions on $U$ by setting $(k.f)(x) := f(k^{-1} \cdot x)$. Similarly one defines a representation of $K$ on $\mathcal{M}(U)$, the space of meromorphic functions on $U$.

A holomorphic function $f$ defined on an open $K$-invariant set $U \subset X$ is called $K$-finite if the set $\{k.f; \ k \in K\}$ is contained in a finitely generated vector subspace of $\mathcal{O}(U)$. For the proof of our main result we need the following (well-known, compare [Ri], sections 2 and 3) fact:

3.1 Lemma. Let $x$ be a point of a $K$-invariant Stein open set $U \subset X$. Then there exist $K$-finite functions $f_1, \ldots, f_r \in \mathcal{O}(U)$ that provide local coordinates at $x$.

Proof. Since $U$ is Stein, we can choose functions $h_1, \ldots, h_r \in \mathcal{O}(U)$ that provide local coordinates at $x$. A result of Harish-Chandra (see [Bo], Chap. 3) implies that the set of the $K$-finite functions of $\mathcal{O}(U)$ is dense in $\mathcal{O}(U)$ with respect to the topology of compact convergence. By replacing each $h_i$ by a sufficiently good $K$-finite approximation $f_i \in \mathcal{O}(U)$ of $h_i$ we obtain the desired result. □

The rest of this section is devoted to give a condition on that locally defined $K$-finite functions can be extended to global meromorphic functions (see Lemma 3).

An open $K$-invariant set $U \subset X$ is called $K$-irreducible, if there exists an irreducible component $U'$ of $U$ such that $U = K \cdot U'$. In particular the induced action of $K$ on the set of the irreducible components of $U$ is transitive.

For the algebra $\mathcal{M}(U)^K$ of $K$-invariant meromorphic functions of a $K$-irreducible open set $U$ we note:

3.2 Lemma. Suppose that $X$ is almost homogeneous with respect to the action of $G$ (i.e., there exists an orbit $W := G \cdot x_0$ that is open and dense in $X$). Then $\mathcal{M}(U)^K \cong \mathbb{C}$ for every $K$-irreducible open set $U \subset X$.

Proof. Let $f \in \mathcal{M}(U)^K$. Since $U$ is $K$-irreducible, it suffices to show that $f$ is constant on a non empty open subset $U_1$ of $U$. We choose a non empty open $U_1 \subset W \cap U$ such that $f$ is holomorphic on $U_1$ and $U_1$ is connected.

Now we fix a point $x \in U_1$ and consider the isomorphism $\varphi : G/G_x \to W$ induced by the orbit map $g \mapsto g \cdot x$. By assumption $f \circ \varphi$ is constant on the (non empty) subset $K G_x \cap \varphi^{-1}(U_1)$ of $G/G_x$. 

3
Since $\varphi^{-1}(U_1)$ is connected, we can apply the identity principle 1.3 of [He] to obtain that $f \circ \varphi$ is constant on all $\varphi^{-1}(U_1)$. Clearly this implies that $f$ is constant on $U_1$. □

3.3 Lemma. Suppose that

i) there exists an open dense orbit $W := G \cdot x_0$ such that the isotropy group of $x_0$ is trivial (in particular $W$ is in a natural way an affine algebraic variety),

ii) every rational function on $W$ can be extended to a meromorphic function on $X$.

Then every $K$-finite function $f$ defined on an open $K$-irreducible set $U \subset X$ can be extended to a meromorphic function on $X$.

Proof. First we consider the complex vector space $V \subset \mathcal{O}(U)$ spanned by $K \cdot f$, where $K$ acts on $\mathcal{O}(U)$ as above. Then $V$ is of finite dimension and the action of $K$ on $V$ is continuous linear.

Next let us consider the complex vector space $\mathbb{C}[W]$ of regular functions of the affine variety $W$. By definition of the affine structure of $W$ (property i)), the action of $G$ on $W$ is algebraic. Thus setting $g \cdot h(x) := h(g^{-1} \cdot x)$ for $h \in \mathbb{C}[W]$ yields a rational representation of $G$ on $\mathbb{C}[W]$. By property ii) we have $\mathbb{C}[W] \subset \mathcal{M}(X) \subset \mathcal{M}(U)$.

Denote by $R \subset \mathcal{M}(U)$ the $\mathbb{C}$-algebra generated by $V$ and $\mathbb{C}[W]$. Then $R$ is finitely generated and $K \cdot R = R$. Moreover the natural $K$-action on $R$ can be uniquely extended to an action of $G$ on $R$ which is linear and rational. Hence $R$ defines an affine $G$-variety $Y := \text{spec}(R)$.

The inclusion $\mathbb{C}[W] \subset R$ gives rise to a $G$-morphism $\varphi : Y \to W$. We claim that $\varphi$ is an isomorphism:

Since $G$ acts freely on $W$, it follows that the restriction of $\varphi$ to any $G$-orbit in $Y$ is an isomorphism. In particular all $G$-orbits of $Y$ have the same dimension. Hence every such orbit is closed in $Y$. Now, by Lemma 2, every $G$-invariant function lying in $R \subset \mathcal{M}(U)$ is constant. But this implies that there exists precisely one closed orbit in $Y$. So we obtain our claim.

In particular this shows that $\mathbb{C}[W] = R$, so one has $f \in \mathbb{C}[W] \subset \mathcal{M}(X)$. □

4 Main Results

4.1 Proposition. Let the normal Moishezon space $X$ be an equivariant compactification of the reductive Lie group $G \cong K^\mathbb{C}$. Suppose that $p \in X$ has an open Stein $K$-invariant neighbourhood $U \subset X$. Then $p$ is schematic.

Proof. First we check that the requirements of Lemma 3 are given. For i) it is sufficient to show that the complement of the open orbit $W$ is an analytic set in $X$. But this is well known (compare for example [Po], Proposition 1).
Property ii) follows from the facts that $X$ is a normal Moishezon space and $G$ acts algebraically: $W$ is an affine Zariski open subset of the algebraic variety $X_{\text{sch}}$ of schematic points of $X$. So we get an isomorphism of the algebras of rational functions $\mathbb{C}(W)$ and $\mathbb{C}(X_{\text{sch}})$. Since $X \setminus X_{\text{sch}}$ is analytic of codimension $\geq 2$ and $X$ is normal, one obtains an embedding $\mathbb{C}(X_{\text{sch}}) \rightarrow \mathcal{M}(X)$.

So in our situation all the assumptions made on $X$ and $G$ in Lemma 3 are valid. According to Lemma 1 there exist $K$-finite functions $f_1, \ldots, f_r \in \mathcal{O}(U)$ that provide local coordinates at $p$. By appropriate shrinking we can achieve that $K \cdot p$ meets every connected component of $U$. Since $X$ is normal, $U$ then has to be $K$-irreducible. So, by Lemma 3, each $f_i$ can be extended to an element of $\mathcal{M}(X)$. □

4.2 Corollary. Suppose that the normal Moishezon space $X$ is an equivariant compactification of the reductive Lie group $G$ and let $p \in X$. If the isotropy group $G_p$ of $p$ is reductive then $p$ is schematic.

Proof. Let $K$ be a maximal compact subgroup of $G$. Since the isotropy group of $p$ is reductive, it follows that there exists a $g \in G$ such that $y := g \cdot p$ is a totally real $K$-point (i.e., the inclusion $K_y \subset G_y$ induces an isomorphism $K^C_y \rightarrow G_y$, see [He;Lo], Section 2.2).

Clearly it suffices to show that $y$ is schematic. But this follows from Proposition 4.1 and the fact that $K \cdot y$ has open Stein $K$-invariant neighbourhoods in $X$ (compare [He;Lo], Section 2.3). □

It should be mentioned, that our criterion for a point to be schematic is not a necessary condition: In the following we construct a smooth algebraic equivariant compactification $X$ of $SL(2, \mathbb{C})$ such that there exist points in $X$ that have no Stein $SU(2)$-invariant neighbourhood.

4.3 Example. Let $B$ denote the Borel subgroup of $SL(2, \mathbb{C})$ consisting of all upper triangular matrices of determinant one. We consider the (algebraic) action of $B$ on $\mathbb{P}_2$ given by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot [z_0, z_1, z_2] := [az_0 + bz_1, a^{-1}z_1, z_2].$$

Then $p_0 := [0, 0, 1]$ is a fixed point of this action and for $p_1 := [1, 1, 1]$ the orbit map $B \rightarrow \mathbb{P}_2, h \mapsto h \cdot p_1$ is an open embedding. We set

$$X := SL(2, \mathbb{C}) \times_B \mathbb{P}_2 := (SL(2, \mathbb{C}) \times \mathbb{P}_2)/B,$$

where $B$ acts on $SL(2, \mathbb{C}) \times \mathbb{P}_2$ by $h \cdot (g, p) := (gh^{-1}, hp)$. Endowed with the $SL(2, \mathbb{C})$-action defined by

$$g_1 \cdot [g, p] := [g_1g, p],$$

5
$X$ is a smooth algebraic equivariant compactification of $SL(2, \mathbb{C})$ (in fact the orbit map $g \mapsto g \cdot [e, p_1]$ is an open embedding with dense image). The orbit

$$A := SL(2, \mathbb{C}) \cdot [e, p_0] \cong SL(2, \mathbb{C})/B$$

is a compact curve in $X$. Now every element $g$ of $SL(2, \mathbb{C})$ can be written as a product $g = uh$ with $u \in SU(2)$ and $h \in B$. Hence it follows that

$$SU(2) \cdot [e, p_0] = A.$$  

Since $A$ is a compact curve in $X$, it cannot be contained in any open Stein neighbourhood of $[e, p_0]$. In particular the point $[e, p_0]$ does not possess a $SU(2)$-invariant open Stein neighbourhood in $X$.

References


