University of Konstanz
Department of Computer & Information Science
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Algorithmic Graph Theory
SS 2012

Assignment 5

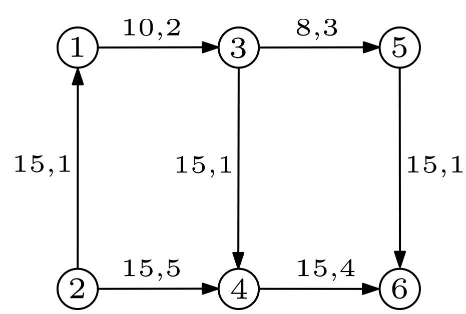
Post Date: 13 June 2012 **Due Date:** 26 June 2012
You are permitted and encouraged to work in groups of two.

Problem 1: Successive Shortest Paths I — 4 Points

Apply the successive shortest path algorithm to the minimum cost flow problem shown below. Explain your solution in detail.

The first number at an arc indicates the upper bound on flow, the second number at an arc indicates the costs per unit of the flow. Further, the vertices have the following demands and supplies, respectively:

$$b(1) = 5, \qquad b(2) = 10$$
$$b(3) = 0, \qquad b(4) = 0$$
$$b(5) = -5, \qquad b(6) = -10$$



Problem 2: Successive Shortest Paths II — 3 Points

Construct a class of minimum cost flow problems for which the number of iterations performed by the successive shortest path algorithm might grow exponentially in $\log U$, where $U = \max(\max_{a \in A} u(a), \max_{v \in V} b(v))$.

[please turn over]

Problem 3: Shortest Paths **4 Points**

Let $\mathcal{N}$ be a network, and let $\mathcal{N}_\pi$ be the network where the arcs have reduced costs c_π. Show that a shortest path from a fixed vertex s to a fixed vertex t in $\mathcal{N}$ is also a shortest path from s to t in $\mathcal{N}_\pi$.

Problem 4: Primal-Dual Algorithm **9 Points**

The primal-dual algorithm works similar to the successive shortest path algorithm. However, instead of sending flow along one shortest path, it solves a maximum flow problem and sends flow along all shortest paths.

Let $\mathcal{N} = (D = (V, A), u, c, b)$ be a network where all arc costs are non-negative. Add a super source s and a super sink t to V. For each vertex $v \in V$ with $b(v) > 0$, add arcs (s, v) with cost zero and capacity $b(v)$ to A. For each vertex $v \in V$ with $b(v) < 0$, add arcs (v, t) with cost zero and capacity $-b(v)$ to A. Set $b(s) = \sum_{\{v \in V | b(v) > 0\}} b(v)$, $b(t) = -b(s)$, and $b(v) = 0$ for all $v \in V$.

The *admissible network* $\mathcal{N}_f^\circ$ is a subgraph of the residual network $\mathcal{N}_f$. It is defined with respect to a pseudoflow f that satisfies the reduced cost optimality condition for some vertex potentials π and contains only those arcs with zero reduced costs. The residual capacity of an arc in $\mathcal{N}_f^\circ$ is the same as in $\mathcal{N}$.

The algorithm now works as follows: Start with zero vertex potentials and zero flow. While $b(s) > 0$, determine the shortest paths from s to all other vertices in $\mathcal{N}_f$ with respect to the reduced costs. Update the vertex potentials $\pi = \pi + \text{dist}(s, v)$, establish a maximum flow $f_{\max}$ from s to t in $\mathcal{N}_f^\circ$, add $f_{\max}$ to f, and update the supply and demand of s and t, respectively.

(a) Show that the potential of a vertex v reflects the distance from v to s in the residual network.

(b) Show that the vertex potential of t strictly increases at each iteration of the algorithm.

(c) Show that if the vertex potential of s is zero, then $n \cdot \max_{\{a \in A\}}(|c(a)|)$ is an upper bound and $-n \cdot \max_{\{a \in A\}}(|c(a)|)$ is a lower bound on the value of any optimal vertex potential. Assume that D contains an uncapacitated directed path between every pair of vertices.