# Network Modeling

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## Outline



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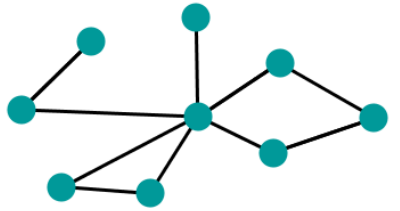
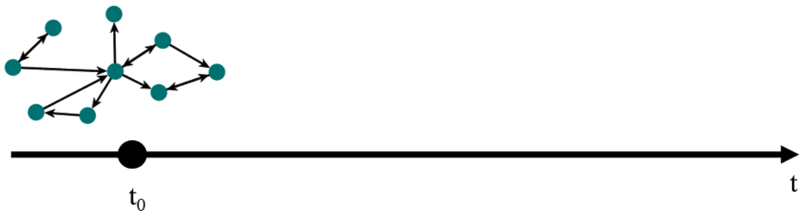
## Where we are



| Model | Main feature | Real data |
|---|---|---|
| $\mathcal{G}(n,p)$ | ties are independent | ties are usually dependent |
| Preferential attachment | based on degree distribution | there are other structural properties |
| ERGM | class of models | reasonable representation of the data |

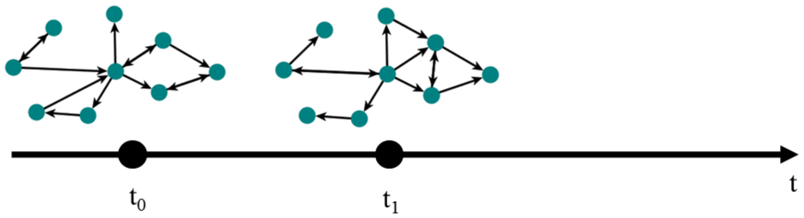
These are models for cross-sectional data

## Where we are going



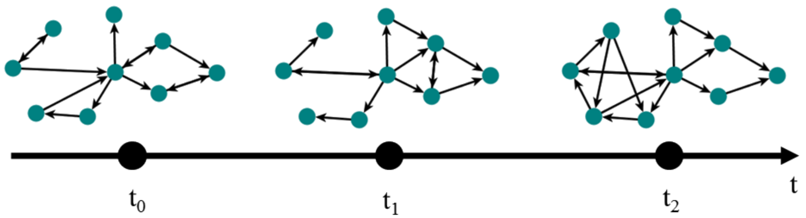



## Where we are going






## Where we are going




Network are dynamic by nature. How to model network evolution?

We need a model for longitudinal data



## Networks are dynamic by nature: a real example

The *Teenage Friends and Lifestyle Study* analyzes smoking behavior and friendship

Data collection: (available from http://www.stats.ox.ac.uk/∼snijders/siena/)

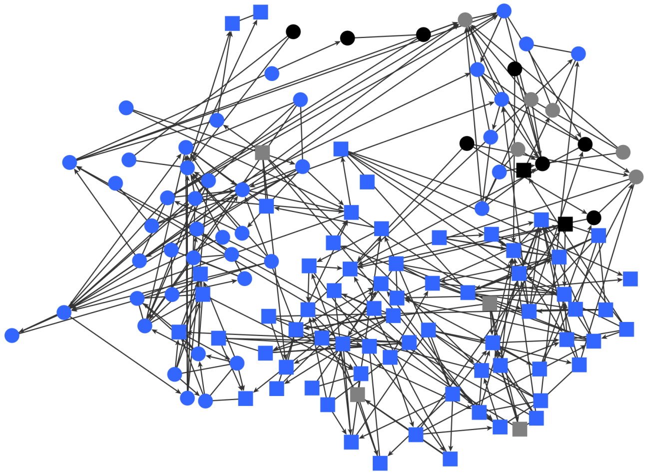
- One school year group monitored over 3 years;
- questionnaires at approximately one year interval:
  1. Friendship relation: each pupil could name up to 12 friends
  2. Individual information and lifestyle elements: gender, age, substances use, smoking of parents and siblings etc.

arrows = friendship relation

gender: circle = girl, square = boy

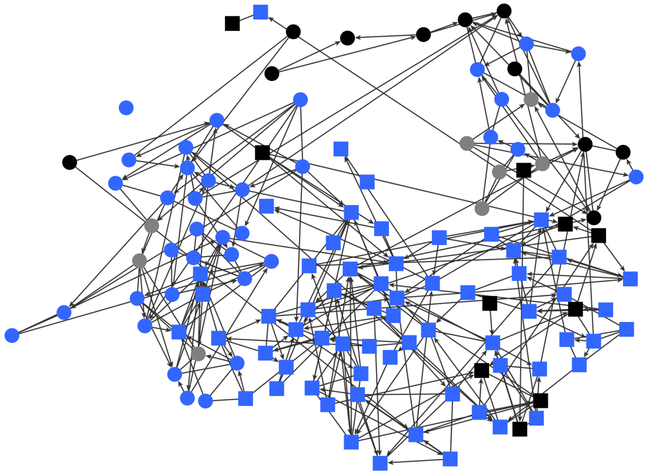
smoking behavior: blue = non, gray = occasional, black = regular

## Networks are dynamic by nature: a real example



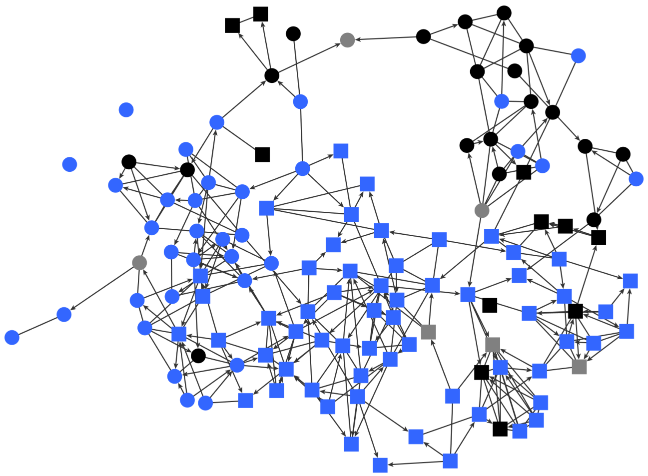


## Networks are dynamic by nature: a real example





## Networks are dynamic by nature: a real example

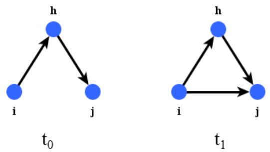




## Some questions

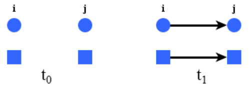
- Is there any tendency in friendship formation towards reciprocity?


- Is there any tendency in friendship formation towards transitivity?

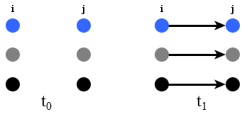


## Some questions

- Is there any homophily in friendship formation with respect to gender?




- Is there any homophily in friendship formation with respect to smoking behavior?






## Solution

Stochastic actor-oriented model (SAOM)

### Aim

Explain network evolution as a result of

- endogenous variables: structural effects depending on the network only (e.g. reciprocity, transitivity, etc.)
- exogenous variables: actor-dependent and dyadic-dependent covariates (e.g. effect of a covariate on the existence of a tie or on homophily)

simultaneously

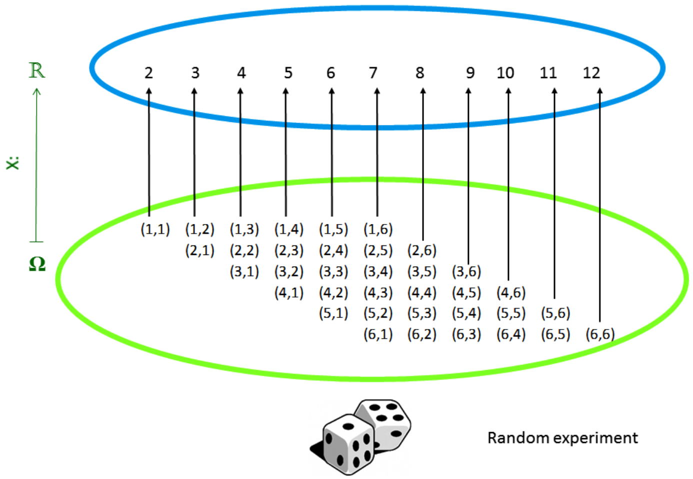


## Background: random variable

### Definition

Let $(\Omega, P)$ be a probability space.
A (real-valued) *random variable* (r.v.) is a function $X : \Omega \to \mathbb{R}$.

### Example



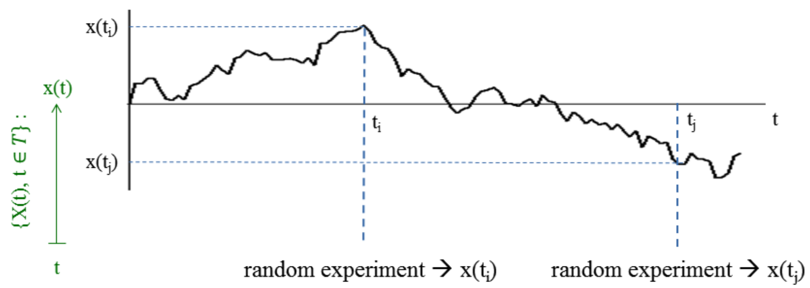



## Background: stochastic (or random) process

### Definition

A stochastic process $\{X(t), t \in \mathcal{T}\}$ is a mapping

$$\forall t \in \mathcal{T} \mapsto X(t) : \Omega \to \mathbb{R}$$



## Background: stochastic process

$\mathcal{T}$ = index set (usually interpreted as time)
$\mathcal{S}$ = state space

Different stochastic processes can be defined according to $\mathcal{S}$ and $\mathcal{T}$

| $\mathcal{S}$ | $\mathcal{T}$ | |
|---|---|---|
| | **Countable** (discrete) | **Uncountable** (continuous) |
| **Countable (finite)** | discrete-time with finite state space | continuous-time with finite state space |
| **Uncountable (continuous)** | discrete-time with continuous state space | continuous-time with continuous state space |

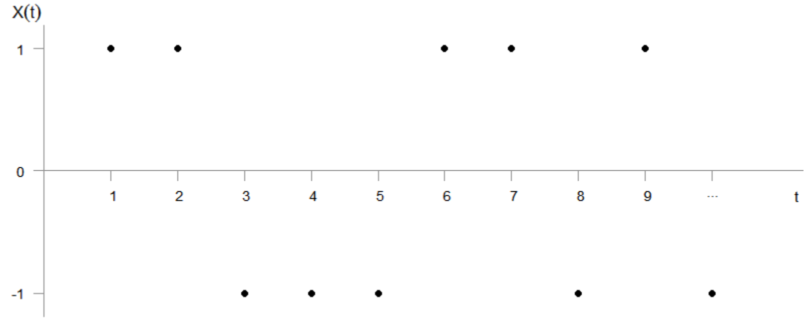


## Background: stochastic process

### Example

$X(t)$ = the outcome of flipping a coin

$\mathcal{S} = \{-1, 1\}$, where $-1$ =tail 1 =head
$\mathcal{T} = \{1, 2, \cdots\}$




$\{X(t), t \in \mathcal{T}\}$ is a discrete-time stochastic process with a finite state space

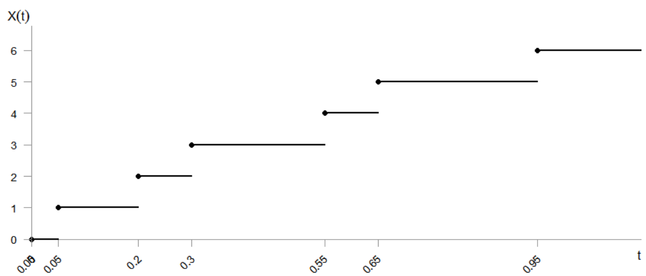


## Background: stochastic process

### Example

$X(t)$ = the number of telephone call at a switchboard of a company from 8 a.m. to 8 p.m.

$\mathcal{S} = \{0, 1, 2, \cdots\}$
$\mathcal{T} = [0, 12]$




$\{X(t), t \in \mathcal{T}\}$ is a continuous-time stochastic process with a finite state space



## Background: continuous-time Markov Chain

### Definition

$\{X(t),\ t \in \mathcal{T}\}$ has the *Markov property* if:

$\forall\ x \in \mathcal{S}$ and $\forall\ t_i < t_j$

$$P(X(t_j) = x(t_j) \mid X(t) = x(t)\ \ \forall\ \ t \leq t_i) = P(X(t_j) = x(t_j) \mid X(t_i) = x(t_i))$$

### Definition

A *continuous-time* Markov chain $\{X_t,\ t \geq 0\}$ is a stochastic process having

1. **finite state**
2. **continuous-time**
3. the **Markovian property**

## Background: continuous-time Markov Chain

### Example

$X(t) = \#$ of goals that a given soccer player scores by time $t$ (time played in official matches)

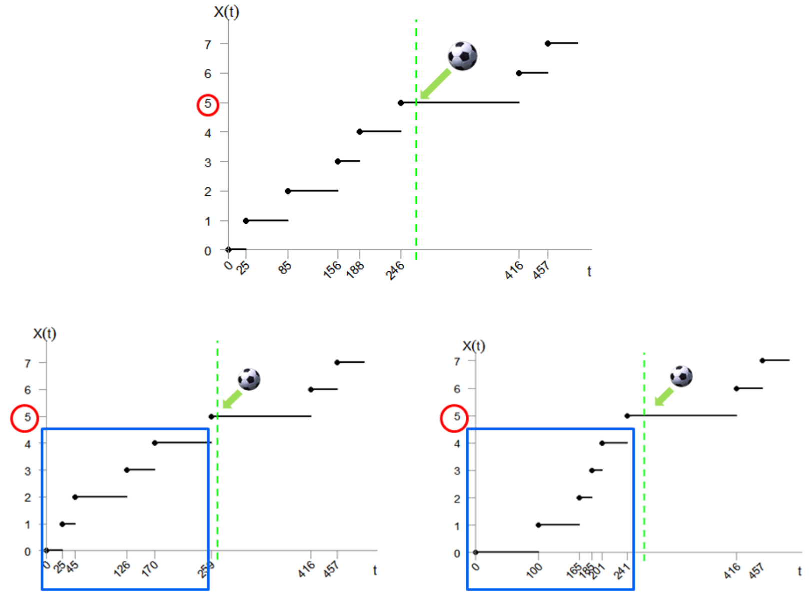
$$\{X(t),\ t \geq 0\} \text{ is a continuous-time Markov chains}$$

Why?

1. state space: $\mathcal{S} = \{0, 1, 2, \ldots, B\}$
   B = total number of goals scored during the career
2. the time is continuous: [0,T]
   T = time of retirement
3. the process $\{X(t),\ t \geq 0\}$ has the Markov property

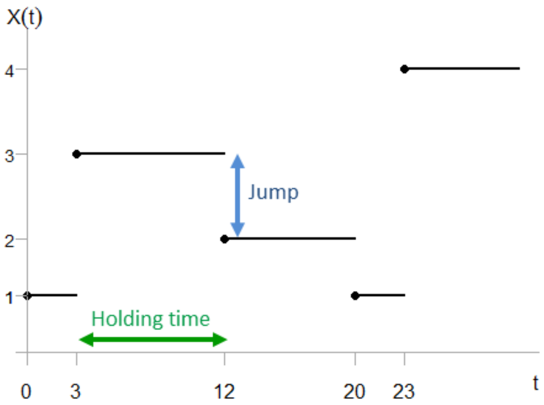


## Background: Markov property






## Background: describing a continuous-time Markov chain






## Background: describing a continuous-time Markov chain

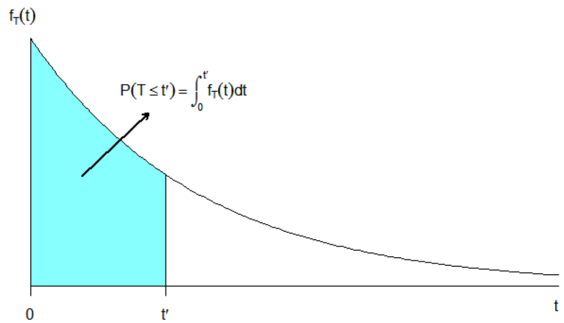
### Holding time

T = amount of time the chain spends in state $i$ (Exponential r.v.)

$$f_T(t) = \lambda_i e^{-\lambda_i t}, \qquad \lambda_i > 0, \quad t > 0$$

$f_T(t) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$P(T \leq t') = \int_0^{t'} f_T(t)dt = 1 - e^{-\lambda_i t'} \qquad \forall t \geq 0$$



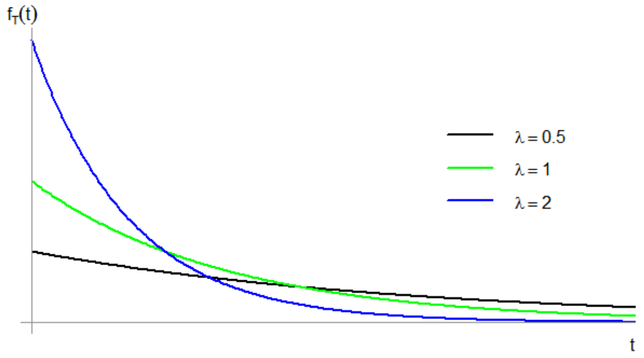
## Background: describing a continuous-time Markov chain

### Holding time

T = amount of time the chain spends in state $i$ (Exponential r.v.)

$$f_T(t) = \lambda_i e^{-\lambda_i t}, \qquad \lambda_i > 0, \quad t > 0$$

$\lambda_i$ is the rate parameter






## Background: describing a continuous-time Markov chain

### Holding time

The Exponential r.v. has the *memoryless property*

$$P(T > s + t \mid T > t) = P(T > s) \quad \forall\, s, t > 0$$

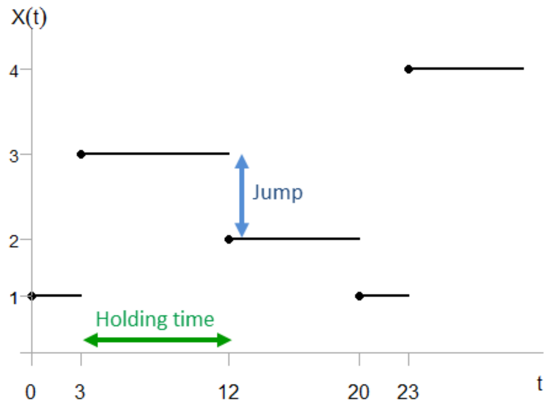
*Proof.*

$$P(T > s + t \mid T > t) = \frac{P(T > t+s \,\cap\, T > t)}{P(T > t)} = \frac{P(T > t+s)}{P(T > t)} =$$

$$= \frac{1 - P(T \leq t+s)}{1 - P(T \leq t)} = \frac{1 - 1 + e^{-\lambda_i (t+s)}}{1 - 1 + e^{-\lambda_i t}} = e^{-\lambda_i s} = P(T > s)$$



## Background: describing a continuous-time Markov chain






## Background: describing a continuous-time Markov chain

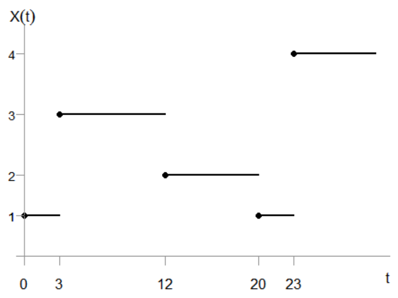
### Jump chain

$P = (p_{ij} : i, j \in \mathcal{S}) =$ *jump matrix*

$$p_{ij} = P(X(t') = j | X(t) = i, \textit{ the opportunity to leave } i)$$

$$p_{ij} \geq 0 \qquad \sum_{j \in \mathcal{S}} p_{ij} = 1 \qquad \forall i, j \in S$$

## Background: describing a continuous-time Markov chain




Example

$$P = \begin{bmatrix} 0.1 & 0 & 0.6 & 0.3 \\ 0.8 & 0.1 & 0.1 & 0 \\ 0.05 & 0.5 & 0.05 & 0.4 \\ 0.6 & 0.1 & 0.15 & 0.15 \end{bmatrix}$$



## Outline

Introduction

The Stochastic actor-oriented model
- Data and model definition
- Model specification
- Parameter interpretation
- Simulating network evolution
- Parameter estimation: MoM and MLE

Extending the model: analyzing the co-evolution of networks and behavior

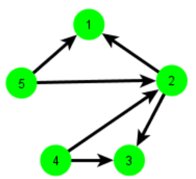
Something more on the SAOM



## Recall: adjacency matrix and directed relations

Social network: a set of actors $\mathcal{N}$ + a relation $\mathcal{R}$ existing among them

**Graph** $= G(\mathcal{N}, \mathcal{R})$




**Adjacency matrix**$=X$

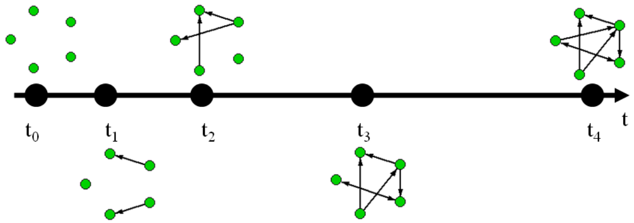
| | | | | |
|---|---|---|---|---|
| - | 0 | 0 | 0 | 0 |
| 1 | - | 1 | 0 | 0 |
| 0 | 0 | - | 0 | 0 |
| 0 | 1 | 1 | - | 0 |
| 1 | 1 | 0 | 0 | - |

Directed relation:


$i \to j \quad \not\Rightarrow \quad j \to i$



## Data




Longitudinal (or panel) network data = M ($\geq 2$) repeated observations on a network

$$x(t_0),\ x(t_1),\ \ldots,\ x(t_m),\ \ldots,\ x(t_{M-1}),\ x(t_M)$$

- set of actors $\mathcal{N} = \{1, 2, \ldots, n\}$
- a non reflexive and directed relation $\mathcal{R}$
- actor covariates $V$ (gender, age,social status, ...)

## Model definition: assumptions

Network evolution is the outcome of a **Continuous-time Markov-Chain**

1. *Ties are state*:
   a tie is a state with a tendency to endure over time

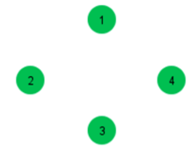
2. *Distribution of the process*:
   $\{X(t), t \in \mathcal{T}\}$ is a continuous time Markov Chain defined on:
   - the state space $\mathcal{X}$
   - the set of actors $\mathcal{N}$



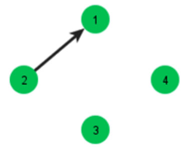
## Model definition: assumptions

**State space**: $\mathcal{X}$ is the set of all possible adjacency matrices defined on $\mathcal{N}$

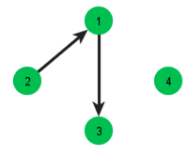
$$\mathcal{X} = 2^{n(n-1)} \Rightarrow \mathcal{X} \textit{ is a countable set}$$



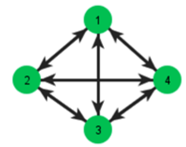

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 0 | 0 | 0 |
| 2 | 0 | - | 0 | 0 |
| 3 | 0 | 0 | - | 0 |
| 4 | 0 | 0 | 0 | - |




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 0 | 0 | 0 |
| 2 | 1 | - | 0 | 0 |
| 3 | 0 | 0 | - | 0 |
| 4 | 0 | 0 | 0 | - |




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 0 | 1 | 0 |
| 2 | 1 | - | 0 | 0 |
| 3 | 0 | 0 | - | 0 |
| 4 | 0 | 0 | 0 | - |

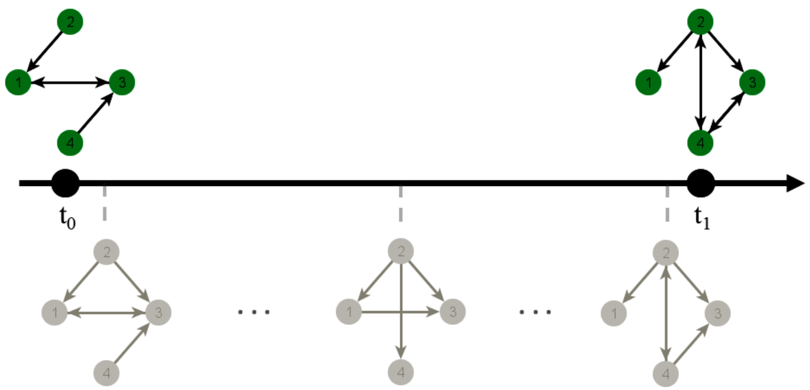



| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 1 | 1 | 1 |
| 2 | 1 | - | 1 | 1 |
| 3 | 1 | 1 | - | 1 |
| 4 | 1 | 1 | 1 | - |



## Model definition: assumptions

**Continuous-time process**




**Latent process**: the network evolves in continuous-time but we observed it only at discrete time points

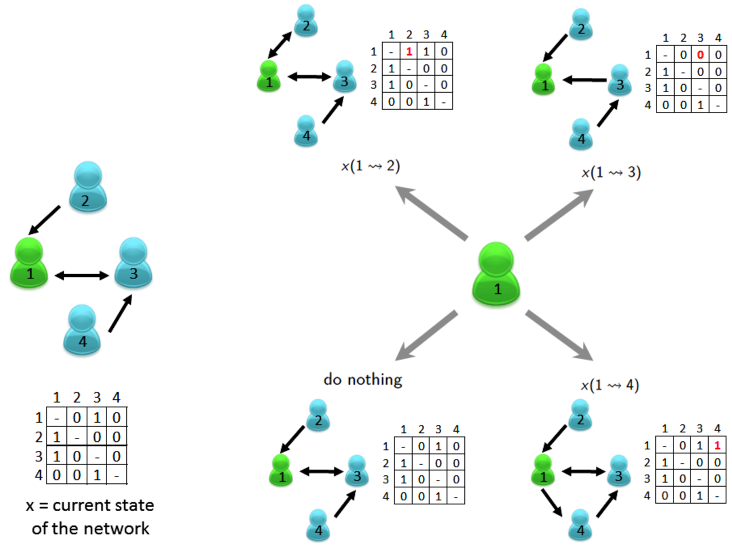


## Model definition: assumptions

**Markov property**: the current state of the network determines probabilistically its further evolution

## Model definition: assumptions

4. *Absence of co-occurrence*: no more than one tie can change at any given moment $t$






## Model definition: assumptions

5. *Actor-oriented perspective*: actors control their outgoing ties

   - change in ties are made by the actor who sends the ties

   - decisions are made according to the position of the actor in the network, his attributes and the characteristics of the others

     **Aim**: maximize *a utility function*

   - actors have complete knowledge about the network and all the other actors

   - the maximization is based on immediate returns (myopic actors)



## Model definition: assumptions (recap)

1. *Ties are states*
2. *The evolution process is a continuous-time Markov chain*
3. *At any given moment t one probabilistically selected actor has the opportunity to change*
4. *No more than one tie can change at any given moment t*
5. *Actor-oriented perspective*



## Model definition

**Consequences of the assumptions**

The evolution process can be decomposed into micro-steps

| Micro-step | Continuous-time Markov chain |
|---|---|
| - the time at which $i$ had the opportunity to change | - the waiting time until the next opportunity for a change made by an actor $i$ (holding time) |
| - the precise change $i$ made | - the probability of changing the link $x_{ij}$ given that $i$ is allowed to change (jump chain) |

Distribution of the holding time: *rate function*

Transition matrix of the jump chain: *objective function*

## Model definition: rate function

**How fast is the opportunity for changing?**

Waiting time between opportunities of change for actor $i \sim Exp(\lambda_i)$

$\lambda_i$ is called the rate function

Simple specification: all actors have the same rate of change $\lambda$

$$P(i \text{ has the opportunity of change}) = \frac{1}{n} \quad \forall i \in \mathcal{N}$$



## Model definition: rate function

**How fast is the opportunity for changing?**

More complex specification

Actors may change their ties at different frequencies $\lambda_i(\alpha, x, v)$

Example

"Young girls might change their ties more frequently"

$$\lambda_i(\alpha, x, v) = \alpha_{age} * v_{age} + \alpha_{gender} * v_{gender}$$

It follows

$$P(i \text{ has the opportunity of change}) = \frac{\lambda_i(\alpha, x, v)}{\sum_{j=1}^{n} \lambda_j(\alpha, x, v)}$$



## Model definition: rate function

**How fast is the opportunity for changing?**

In the following we assume that:

- all actors have the same rate of change

  $\Longrightarrow$ $\lambda$ is constant over the actors

- the frequencies at which actors have the opportunity to make a change depends on time

  $\Longrightarrow$ $\lambda$ is not constant over time

As a consequence, we must specify $M-1$ rate functions

$$\lambda_1, \cdots, \lambda_{M-1}$$



## Model definition: objective function

**Which tie is changed?**

Changing a tie means turning it into its opposite:

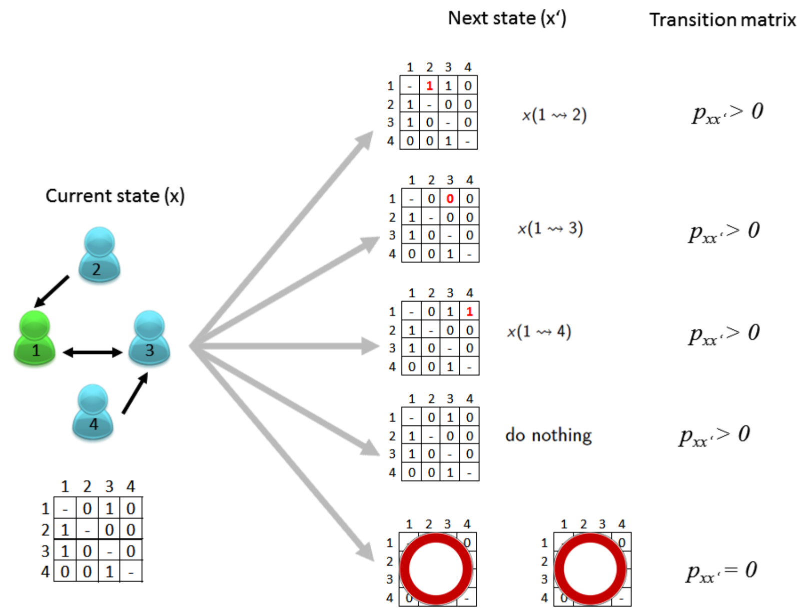
$x_{ij} = 0$ is changed into $x_{ij} = 1$ tie creation

$x_{ij} = 1$ is changed into $x_{ij} = 0$ tie deletion

Given that $i$ has the opportunity to change:

| Possible choices of $i$ | Possible reachable states |
|---|---|
| $n-1$ changes | $n-1$ networks $x(i \rightsquigarrow j)$ |
| 1 non-change | 1 network equal to x |

## Model definition: objective function






## Background: random utility model

Setting: decision makers who face a choice between $n$-alternatives

Decision rule: choose the alternative that assures the highest utility

$$U_{ij} = F_{ij} + \mathcal{E}_{ij}$$

$F_{ij}$: deterministic part of the utility

$\mathcal{E}_{ij}$: random term

Decision probabilities: for a suitable choice of $\mathcal{E}_{ij}$

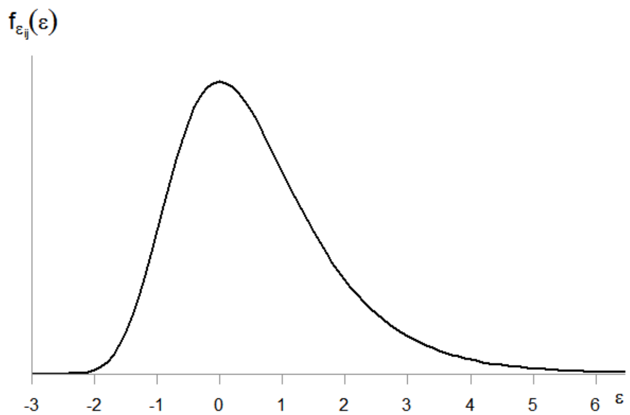
$$p_{ij} = \frac{e^{F_{ij}}}{\sum_{j=1}^{n} e^{F_{ij}}}$$



## Background: random utility model

It is assumed that $\mathcal{E}_{ij}$ is Gumbel distributed

$$f_{\mathcal{E}_{ij}}(\epsilon) = e^{-\epsilon} e^{-e^{-\epsilon}} \qquad \epsilon \in \mathbb{R}$$






## Model definition: objective function

Actors change their ties in order to maximize a utility function

$$u_i(\beta, x(i \rightsquigarrow j)) = f_i(\beta, x(i \rightsquigarrow j)) + \mathcal{E}_i(t, x, j)$$

- $f_i(\beta, x(i \rightsquigarrow j))$ is the *objective function*
- $\mathcal{E}_i(t, x, j))$ is a random utility term Gumbel distributed

Probabilities

$$p_{ij} = \frac{exp\left(f_i(\beta, x(i \rightsquigarrow j))\right)}{\sum_{h=1}^{n} exp\left(f_i(\beta, x(i \rightsquigarrow h)\right))}$$

Probabilities interpretation:

$p_{ij}$ is the probability that $i$ changes the tie towards $j$

$p_{ii}$ is the probability of not changing

## Model definition: objective function

The objective function is defined as a linear combination

$$f_i(\beta, x(i \rightsquigarrow j)) = \sum_{k=1}^{K} \beta_k s_{ik}(x(i \rightsquigarrow j))$$

- $s_{ik}(x(i \rightsquigarrow j))$ are effects
- $\beta_k$ are statistical parameters

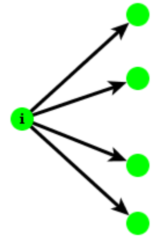


## Objective function specification

***Endogenous effects*** = *dependent on the network structures*

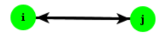
- Outdegree effect

$$s_{i\_out}(x') = \sum_j x'_{ij}$$




- Reciprocity effect

$$s_{i\_rec}(x') = \sum_j x'_{ij} x'_{ji}$$



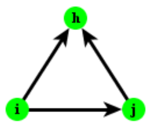



## Objective function specification

***Endogenous effects*** = *dependent on the network structures*

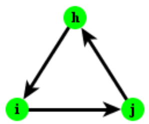
- Transitive effect

$$s_{i\_trans}(x') = \sum_{j,h} x'_{ij} x'_{ih} x'_{jh}$$




- three cycle-effect

$$s_{i\_cyc}(x') = \sum_{j,h} x'_{ij} x'_{jh} x'_{hi}$$

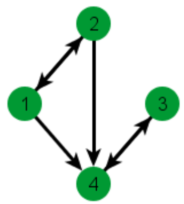


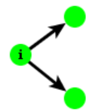
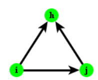



## Objective function specification

Example

$$\beta_{out} = -1 \qquad \beta_{rec} = +0.5 \qquad \beta_{trans} = -0.25$$




| $i \to j$ | $s_{i\_out}$  | $s_{i\_rec}$  | $s_{i\_trans}$  | $f_i$ |
|---|---|---|---|---|
| $1 \to 1$ | 2 | 1 | 1 | -1.75 |
| $1 \to 2$ | 1 | 0 | 0 | -1 |
| $1 \to 3$ | 3 | 1 | 3 | -3.25 |
| $1 \to 4$ | 1 | 1 | 0 | -0.5 |

$$p_{11} = 0.146 \qquad p_{12} = 0.310 \qquad p_{13} = 0.033 \qquad p_{14} = 0.511$$

## Objective function specification

***Exogenous effects*** = *related to actor's attributes*

### Example

- Friendship among pupils:
  Smoking: non, occasional, regular
  Gender: boys, girls

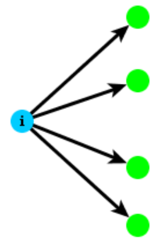
- Trade/Trust (Alliances) among countries:
  Geographical area: Europe, Asia, North-America,...
  Worlds: first, Second, Third, Fourth



## Objective function specification

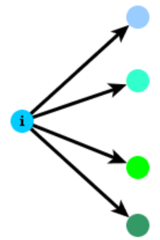
- covariate-ego

$$s_{i\_cego}(x) = \sum_j x_{ij} v_i$$




- covariate-alter

$$s_{i\_calt}(x) = \sum_j x_{ij} v_j$$

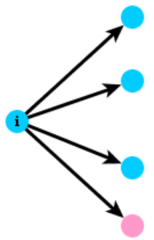





## Objective function specification

- covariate-related similarity

$$s_{i\_csim}(x) = \sum_j x_{ij} \left(1 - \frac{|v_i - v_j|}{R_V}\right)$$




where $R_V$ is the range of $V$ and $\left(1 - \frac{|v_i - v_j|}{R_V}\right)$ is called *similarity score*

Remark:

when $V$ is a binary covariate, the covariate-related similarity can be written in the following way:

$$s_{i\_csim}(x) = \sum_j x_{ij} \mathbb{I}\{v_i = v_j\}$$



## Objective function specification

Which effects must be included in the objective function?

Outdegree and Reciprocity must always be included.
The choice of the other effects must be determined according to hypotheses derived from theory

### Example

Friendship network

| Theory | | Effect |
|---|---|---|
| the friend of my friend is also my friend | $\Rightarrow$ | transitive effect |
| girls trust girls<br>boys trust boys | $\Rightarrow$ | covariate-related similarity |

## Objective function specification

1. Parameter interpretation: $\beta_k$ quantifies the role of $s_{ik}(x')$ in the network evolution.
   - $\beta_k = 0$: $s_{ik}(x')$ plays no role in the network dynamics
   - $\beta_k > 0$: higher probability of moving into networks where $s_{ik}(x')$ is higher
   - $\beta_k < 0$: higher probability of moving into networks where $s_{ik}(x')$ is lower
2. The preferences driving the choice of the actors have the same intensities over time

$$\Longrightarrow \quad \beta_1, \cdots, \beta_K \text{ are constant over time}$$



## Parameter interpretation

The procedures for estimating the parameters of the SAOM are implemented in a R library called ***RSiena***

(SIENA = **S**imulation **I**nvestigation for **E**mpirical **N**etwork **A**nalysis)

The R script "estimation.R" contains the R commands to implement the estimation procedure in R and the folder "tfls.zip" includes the data files.

Example data: an excerpt from the "Teenage Friends and Lifestyle Study" data set:

- Networks: relation = friendship
  actors = 129 pupils present at all three measurement points
- Covariates: gender (1 = Male, 2 = Female)
  smoking behavior (1 = no, 2= occasional, 3 = regular)



## Parameter interpretation: a very simple model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 8.5948 | ( 0.7091 ) | |
| Rate parameter period 2 | 7.2115 | ( 0.5751 ) | |
| *Other parameters:* | | | |
| outdegree (density) | -2.4147 | ( 0.0387 ) | -62.3875 |
| reciprocity | 2.7106 | ( 0.0811 ) | 33.4061 |

Rate parameter: expected frequency, between two consecutive network observations, with which actors get the opportunity to change a network tie

- about 9 opportunities for change in the first period
- about 7 opportunities for change in the second period

The estimated rate parameters will be higher than the observed number of changes per actor (why?)



## Parameter interpretation: a very simple model

Interpreting the objective function parameters:

The parameter $\beta_k$ quantifies the role of the effect $s_{ik}$ in the network evolution.

$\beta_k = 0$ $s_{ik}$ plays no role in the network dynamics

$\beta_k > 0$ higher probability of moving into networks where $s_{ik}$ is higher

$\beta_k < 0$ higher probability of moving into networks where $s_{ik}$ is lower


Which $\beta_k$ are "significantly" different from 0?

E.g. $\beta_{rec} = 0.13$ is "significantly" different from 0?

## Parameter interpretation: a very simple model

Hypothesis test:

1. State the hypotheses.
   - The *null hypothesis* ($H_0$) states that the observed increase or decrease in the number of network configurations related to a certain effect results purely from chance.

$$H_0 : \beta_k = 0$$

   - The *alternative hypothesis* ($H_1$) states that the observed increase or decrease in the number of network configurations related to a certain effect is influenced by some non-random cause.

$$H_1 : \beta_k \neq 0$$



## Parameter interpretation: a very simple model

Hypothesis test:

2. Define a decision rule

$$\begin{cases} \left|\frac{\beta_k}{s.e.(\beta_k)}\right| \geq 2 & \textit{reject } H_0 \\ \left|\frac{\beta_k}{s.e.(\beta_k)}\right| < 2 & \textit{fail to reject } H_0 \end{cases}$$

The logic behind this decision rule is based on the standard error concept.

Example

Is the value $\beta_{rec} = 0.13$ far enough from 0?

If $s.e.(\beta_{rec}) = 0.9$, a more or less plausible set of values that the parameter can assume is approximately

$$[0.04, 0.22]$$

$$\left|\frac{\beta_{rec}}{s.e.(\beta_{rec})}\right| = \left|\frac{0.13}{0.9}\right| = 0.14 < 2$$

$\beta_{rec}$ is not significantly different from 0



## Parameter interpretation: a very simple model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 8.5948 | ( 0.7091 ) | |
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| *Other parameters:* | | | |
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Objective function parameters:

- outdegree parameter: the observed networks have low density



## Parameter interpretation: a very simple model

| | Estimates | s.e. | t-score |
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| *Rate parameters:* | | | |
| Rate parameter period 1 | 8.5948 | ( 0.7091 ) | |
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| *Other parameters:* | | | |
| outdegree (density) | -2.4147 | ( 0.0387 ) | -62.3875 |
| reciprocity | 2.7106 | ( 0.0811 ) | 33.4061 |

Objective function parameters:

- outdegree parameter: the observed networks have low density
- reciprocity parameter: strong tendency towards reciprocated ties

## Parameter interpretation: a very simple model

In more detail

$$\beta_{out}\sum_{j=1}^{n} x_{ij} + \beta_{rec}\sum_{j=1}^{n} x_{ij}x_{ji} = -2.4147\sum_{j=1}^{n} x_{ij} + 2.7106\sum_{j=1}^{n} x_{ij}x_{ji}$$

Adding a reciprocated tie (i.e., for which $x_{ji} = 1$) gives

$$-2.4147 + 2.7106 = 0.2959$$

while adding a non-reciprocated tie (i.e., for which $x_{ji} = 0$) gives

$$-2.4147$$

Conclusion: reciprocated ties are valued positively and non-reciprocated ties are valued negatively by actors



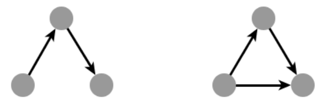
## Parameter interpretation: a more complex model

### Specifying the objective function

In friendship context, sociological theory suggests that:

- friendship relations tend to be reciprocated → reciprocity effect

- the statement "the friend of my friend is also my friend" is almost always true → transitive triplets effect





## Parameter interpretation: a more complex model

### Specifying the objective function

In friendship context, sociological theory suggests that:

- pupils prefer to establish friendship relations with others that are similar to themselves → covariate similarity

This effect must be controlled for the sender and receiver effects of the covariate.

- Covariate ego effect

- Covariate alter effect



## Parameter interpretation: a more complex model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 10.6809 | ( 1.0425 ) | |
| Rate parameter period 2 | 9.0116 | ( 0.8386 ) | |
| | | | |
| *Other parameters:* | | | |
| outdegree (density) | -2.8597 | ( 0.0608 ) | -47.0288 |
| reciprocity | 1.9855 | ( 0.0876 ) | 22.6765 |
| transitive triplets | 0.4480 | ( 0.0257 ) | 17.4558 |
| sex alter | -0.1513 | ( 0.0980 ) | -1.5445 |
| sex ego | 0.1571 | ( 0.1072 ) | 1.4659 |
| sex similarity | 0.9191 | ( 0.1076 ) | 8.5440 |
| smoke alter | 0.1055 | ( 0.0577 ) | 1.8272 |
| smoke ego | 0.0714 | ( 0.0623 ) | 1.1469 |
| smoke similarity | 0.3724 | ( 0.1177 ) | 3.1647 |

- outdegree parameter: the observed networks have low density
- reciprocity parameter: strong tendency towards reciprocated ties
- transitivity parameter: preference for being friends with friends'friends

## Parameter interpretation: a more complex model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 10.6809 | ( 1.0425 ) | |
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| smoke ego | 0.0714 | ( 0.0623 ) | 1.1469 |
| smoke similarity | 0.3724 | ( 0.1177 ) | 3.1647 |

- sex alter: gender does not affect actor popularity
- sex ego: gender does not affect actor activity
- sex similarity: tendency to choose friends with the same gender



## Parameter interpretation: a more complex model

- Gender: coded with 1 for boys and with 2 for girls.
- All actor covariates are centered: $\overline{v} = 1.434$ is the mean of the covariate

$$v_i - \overline{v} = \begin{cases} -0.434 & \textit{for boys} \\ 0.566 & \textit{for girls} \end{cases}$$

- The contribution of $x_{ij}$ to the objective function is

$$\beta_{ego}(v_i - \overline{v}) + \beta_{alter}(v_j - \overline{v}) + \beta_{same}\left(\mathbb{I}\{v_i = v_j\} - sim_v\right) =$$

$$= 0.1571(v_i - \overline{v}) - 0.1513(v_j - \overline{v}) + 0.9191\left(\mathbb{I}\{v_i = v_j\} - 0.5048\right)$$

where $sim_v$ is the average of the similarity score.



## Parameter interpretation: a more complex model

| | Male | Female |
|---|---|---|
| Male | 0.4526 | -0.618 |
| Female | -0.309 | 0.4584 |

Table : Gender-related contributions to the objective function

Conclusions: Preference for intra-gender relationships.



## Parameter interpretation: a more complex model


| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 10.6809 | ( 1.0425 ) | |
| Rate parameter period 2 | 9.0116 | ( 0.8386 ) | |
| *Other parameters:* | | | |
| outdegree (density) | -2.8597 | ( 0.0608 ) | -47.0288 |
| reciprocity | 1.9855 | ( 0.0876 ) | 22.6765 |
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| sex similarity | 0.9191 | ( 0.1076 ) | 8.5440 |
| smoke alter | 0.1055 | ( 0.0577 ) | 1.8272 |
| smoke ego | 0.0714 | ( 0.0623 ) | 1.1469 |
| smoke similarity | 0.3724 | ( 0.1177 ) | 3.1647 |


- smoke alter: smoking behavior does not affect actor popularity
- smoke ego: smoking behavior not affect actor activity
- smoke similarity: tendency to choose friends with the same smoking behavior

## Parameter interpretation: a more complex model

- Smoking behavior: coded with 1 for "no", 2 for "occasional", and 3 for "regular" smokers.
- The smoking covariate is centered: $\overline{v} = 1.310$ is the mean of the covariate

$$v_i - \overline{v} = \begin{cases} -0.310 & \textit{for no smokers} \\ 0.690 & \textit{for occasional smokers} \\ 1.690 & \textit{for regular smokers} \end{cases}$$

- The contribution of $x_{ij}$ to the objective function is

$$\beta_{ego}(v_i - \overline{v}) + \beta_{alter}(v_j - \overline{v}) + \beta_{same}\left(1 - \frac{|v_i - v_j|}{R_v} - sim_v\right) =$$

$$= 0.0714(v_i - \overline{v}) + 0.1055(v_j - \overline{v}) + 0.3724\left(1 - \frac{|v_i - v_j|}{2} - 0.7415\right)$$



## Parameter interpretation: a more complex model

| | no | occasional | regular |
|---|---|---|---|
| no | 0.0414 | -0.0734 | -0.1882 |
| occasional | -0.0393 | 0.2183 | 0.1035 |
| regular | -0.1200 | 0.1376 | 0.3952 |

Table : Smoking-related contributions to the objective function

Conclusions:

- preference for similar alters
- this tendency is strongest for high values on smoking behavior



## Simulating network evolution

Reproducing a possible series of micro-steps between $t_0$ and $t_1$ according to fixed parameter value and the network $x(t_0)$.

$t$ = time
$dt$ = holding time between consecutive opportunities to change

**Algorithm 1:** Network evolution

```
Input: x(t0), λ, β, n
Output: x^sim(t1)
t ← 0
x ← x(t0)
while condition = TRUE do
    dt ~ Exp(nλ)
    i ~ Uniform(1, ..., n)
    j ~ Multinomial(p_i1, ..., p_in)
    if i ≠ j then
        x ← x(i ⇝ j)
    else
        x ← x
    t ← t + dt
x^sim(t1) ← x
return x^sim(t1)
```



## Simulating network evolution

Two different stopping rules:

1. *Unconditional* simulation:

   the simulation of the network evolution carries on until a predetermined time length has elapsed (usually until $t = 1$).

2. *Conditional* simulation on the observed number of changes:

   simulation runs on until the number of different entries between $x(t_0)$ and the simulated network $x^{sim}(t_1)$ is equal to the number of entries that differ between $x(t_0)$ and $x(t_1)$

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left| x_{ij}^{obs}(t_1) - x_{ij}(t_0) \right| = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left| x_{ij}^{sim}(t_1) - x_{ij}(t_0) \right|$$

This criterion can be generalized conditioning on any other explanatory variable.

## Parameter estimation

The formulation of the SAOM depends on $M-1+K$ statistical parameters

$$\theta = (\lambda_1, \cdots, \lambda_{M-1}, \beta_1, \cdots, \beta_K)$$

Aim: estimate $\theta$

Different estimation methods:
- the Method of Moments (MoM)
- the Maximum Likelihood Estimation (MLE)



## Background: Method of Moments (MoM)

Let
- $X$ be a r.v. with distribution $f_X(x;\theta)$
- $E_\theta[X]$ be the expected value of $X$
- $(x_1, \ldots, x_n)$ be $n$ observations from the r.v. $X$.

### Definition
The sample counterpart of $E_\theta[X]$ is defined as:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$



## Background: Method of Moments (MoM)

One can observe that the expected value of a certain distribution usually depends on the parameter $\theta$

### Definition

The method of moment estimator for $\theta$ is found by equating the expected value $E_\theta[X]$ to its sample counterpart $\mu$

$$E_\theta[X] = \mu$$

and solving the resulting equation for the unknown parameter.

In practice:
1. Compute the expected value $E_\theta[X]$
2. Compute the sample counterpart $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$
3. Solve the moment equation $E_\theta[X] = \mu$ for $\theta$



## Background: Method of Moments (MoM)

### Example
10 undirected, simple, loopless graphs are generated according to $\mathcal{G}(30, p)$.
The number of edges $y_i$ in each graph is reported in the following table:

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_i$ | 37 | 40 | 35 | 32 | 39 | 34 | 25 | 28 | 41 | 32 |

Find a plausible value for $p$ that might have generated the observed graphs.

$Y$ = r.v. describing the number of edges

$$P(Y=y) = \binom{N}{y} p^y (1-p)^{N-y}$$

where $N = \frac{n(n-1)}{2}$

## Background: Method of Moments (MoM)

Example

1. The theoretical expected value of the number of edges is:

$$\begin{aligned}
E_\theta[Y] &= \sum_{y=0}^{N} yP(Y=y) = \sum_{y=0}^{N} y\binom{N}{y}p^y(1-p)^{N-y} \\
&= \sum_{y=1}^{N} y\binom{N}{y}p^y(1-p)^{N-y} \\
&= \sum_{y=1}^{N} y\frac{N!}{y!(N-y)!}p^y(1-p)^{N-y} \\
&= Np\sum_{y=1}^{N} \frac{(N-1)!}{(y-1)!(N-1-(y-1))!}p^{y-1}(1-p)^{N-1-(y-1)} \\
&= Np\sum_{i=0}^{N-1}\binom{N-1}{i}p^i(1-p)^{N-1-i} = Np
\end{aligned}$$



## Background: Method of Moments (MoM)

Example

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_i$ | 37 | 40 | 35 | 32 | 39 | 34 | 25 | 28 | 41 | 32 |

2. The sample counterpart is:

$$\mu = \frac{1}{10}\sum_{i=1}^{10} y_i = \frac{343}{10} = 34.3$$

3. The estimate for $p$ is given by:

$$E_\theta[Y] = \mu$$
$$Np = \mu$$
$$\widehat{p} = \frac{\mu}{N} = \frac{34.3}{435} = 0.079$$



## Background: Method of Moments (MoM)

Remark

It is easy to imagine that the estimate of the parameter can vary according to the selected sample.

Example

Other 10 generated graphs result in the following number of edges:

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_i$ | 34 | 28 | 23 | 31 | 32 | 36 | 33 | 41 | 26 | 39 |

The new estimate for $p$ is now

$$\widehat{p} = 0.074$$

This value is close to the one obtained before, but it is not the same!
For this reason, we usually associate to an estimator its standard error.



## Background: Generalizations of MoM

The principle of the MoM can be easily generalized to any function $s(X)$.

1. Expected value of $s(X)$:

$$E_\theta[s(X)] = \sum_x s(x)f_X(x;\theta)$$

2. Corresponding sample moment:

$$\gamma = \frac{1}{n}\sum_{i=1}^{n} s(x_i)$$

3. Moment equation:

$$E_\theta[s(X)] = \gamma$$

The functions $s(X)$ are called *statistics*

## Background: Generalizations of MoM

The MoM can be applied also in situations where $\theta = (\theta_1, \ldots, \theta_p)$.

1. Definition of $p$ statistics $(s_1(X), \ldots, s_p(X))$
2. Definition of $p$ moment conditions:

$$E_\theta[s_1(X)] = \gamma_1$$
$$E_\theta[s_2(X)] = \gamma_2$$
$$\ldots$$
$$E_\theta[s_p(X)] = \gamma_p$$

3. Solving the resulting equations for the unknown parameters



## Estimating the parameter of the SAOM using MoM

Aim: estimate $\theta$ using the MoM

$$\theta = (\lambda_1, \ldots, \lambda_{M-1}, \beta_1, \ldots, \beta_K)$$

In practice:

1. find $M - 1 + K$ statistics
2. set the theoretical expected value of each statistic equal to its sample counterpart
3. solve the resulting system of equations with respect to $\theta$.

For simplicity, let us assume to have observed a network at two time points $t_0$ and $t_1$ and to condition the estimation on the first observation $x(t_0)$



## 1. Defining the statistics

The rate parameter $\lambda$ describes the frequency at which changes can potentially happen.

$$s_\lambda(X(t_1), X(t_0)|X(t_0) = x(t_0)) = \sum_{i,j=1}^{n} \left|X_{ij}(t_1) - X_{ij}(t_0)\right|$$

Reason

| | $\lambda = 2$ | $\lambda = 3$ | $\lambda = 4$ |
|---|---|---|---|
| $s_\lambda$ | 94 | 135 | 171 |

$\Rightarrow$ higher values of $\lambda$ leads to higher values of $s_\lambda$



## 1. Defining the statistics

The parameter $\beta_k$ quantifies the role played by each effect in the network evolution.

$$s_k(X(t_1)|X(t_0) = x(t_0)) = \sum_{i=1}^{n} s_{ik}(X(t_1))$$

Example

Let us consider the outdegree:

$$s_{out}(X(t_1)|X(t_0) = x(t_0)) = \sum_{i=1}^{n} s_{i\_out}(X(t_1)) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}(t_1)$$

| | $\beta_{out} = -2.5$ | $\beta_{out} = -2$ | $\beta_{out} = -1.5$ |
|---|---|---|---|
| $s_{out}$ | 195 | 214 | 234 |

$\Rightarrow$ higher values of $\beta_{out}$ leads to higher values of $s_{out}$

## 1. Defining the statistics

Generalizing to $M-1$ periods:

- Statistics for the rate function parameters

$$s_{\lambda_1}(X(t_1), X(t_0) | X(t_0) = x(t_0)) = \sum_{i,j=1}^{n} \left| X_{ij}(t_1) - X_{ij}(t_0) \right|$$

$$\cdots$$

$$s_{\lambda_{M-1}}(X(t_M), X(t_{M-1}) | X(t_{M-1}) = x(t_{M-1})) = \sum_{i,j=1}^{n} \left| X_{ij}(t_M) - X_{ij}(t_{M-1}) \right|$$

- Statistics for the objective function parameters:

$$\sum_{m=1}^{M-1} s_{mk}\left(X(t_m) | X(t_{m-1}) = x(t_{m-1})\right) = \sum_{m=1}^{M-1} s_{mk}(X(t_m))$$



## 2. Setting the moment equations

The MoM estimator for $\theta$ is defined as the solution of the system of $M+K-1$ equations

$$\begin{cases} E_\theta \left[ s_{\lambda_m}(X(t_m), X(t_{m-1}) | X(t_{m-1}) = x(t_{m-1})) \right] = s_{\lambda_m}(x(t_m), x(t_{m-1})) \\ E_\theta \left[ \sum_{m=1}^{M-1} s_{mk}\left(X(t_m) | X(t_{m-1}) = x(t_{m-1})\right) \right] = \sum_{m=1}^{M-1} s_{mk}(x(t_m)) \end{cases}$$

with $m = 1, \ldots, M-1$ and $k = 1, \cdots, K$



## 2. Setting the moment equations

### Example

Let us assume to have observed a network at $M = 2$ time points


$t_0$:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 1 | 1 | 0 |
| 2 | 1 | - | 0 | 0 |
| 3 | 0 | 0 | - | 1 |
| 4 | 0 | 0 | 1 | - |

$t_1$:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 1 | 0 | 1 |
| 2 | 0 | - | 0 | 0 |
| 3 | 0 | 0 | - | 1 |
| 4 | 1 | 1 | 1 | - |

We want to model the network evolution according to the outdegree, the reciprocity and the transitivity effects

$$\theta = (\lambda, \beta_{out}, \beta_{rec}, \beta_{trans})$$



## 2. Setting the moment equations

### Example

Statistics:

$$s_\lambda(X(t_1), X(t_0) | X(t_0) = x(t_0)) = \sum_{i,j=1}^{4} \left| X_{ij}(t_1) - X_{ij}(t_0) \right|$$

$$s_{out}\left(X(t_1) | X(t_0) = x(t_0)\right) = \sum_{i,j=1}^{4} X_{ij}(t_1)$$

$$s_{rec}\left(X(t_1) | X(t_0) = x(t_0)\right) = \sum_{i,j=1}^{4} X_{ij}(t_1) X_{ji}(t_1)$$

$$s_{trans}\left(X(t_1) | X(t_0) = x(t_0)\right) = \sum_{i,j,h=1}^{4} X_{ij}(t_1) X_{ih}(t_1) X_{jh}(t_1)$$

## 2. Setting the moment equations

### Example


| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 1 | 1 | 0 |
| 2 | 1 | - | 0 | 0 |
| 3 | 0 | 0 | - | 1 |
| 4 | 0 | 0 | 1 | - |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 1 | 0 | 1 |
| 2 | 0 | - | 0 | 0 |
| 3 | 0 | 0 | - | 1 |
| 4 | 1 | 1 | 1 | - |

Observed values of the statistics:

$$s_\lambda = 5$$

$$s_{out} = 6 \quad s_{rec} = 4 \quad s_{trans} = 2$$



## 2. Setting the moment equations

### Example

We look for the value of $\theta$ that satisfies the system:

$$\begin{cases} E_\theta \left[ s_\lambda (X(t_1), X(t_0) | X(t_0) = x(t_0)) \right] = 5 \\ E_\theta \left[ s_{out} \left( X(t_1) | X(t_0) = x(t_0) \right) \right] = 6 \\ E_\theta \left[ s_{rec} \left( X(t_1) | X(t_0) = x(t_0) \right) \right] = 4 \\ E_\theta \left[ s_{trans} \left( X(t_1) | X(t_0) = x(t_0) \right) \right] = 2 \end{cases}$$



## 3. Solving the moment equations

Simplified notation:

- S: $(M - 1 + K)$-dimensional vector of statistics
- s: $(M - 1 + K)$-dimensional vector of the observed values of the statistics

Consequently, the system of moment equations can be written as

$$E_\theta[S] = s$$

or equivalently as

$$E_\theta[S - s] = 0$$

Problem:
analytical procedures cannot be applied to solve this system



## 3. Solving the moment equations

**Definition**
*Stochastic approximation methods* are a family of iterative stochastic algorithms that attempt to find zeros of functions which cannot be analytically computed.

The Robbins-Monro (RM) algorithm: iterative algorithm to find the solution to

$$E_\theta[X] = \alpha$$

The value of $\theta$ is iteratively updated according to:

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i \left( E_{\widehat{\theta}_i}[X] - \alpha \right)$$

where

$$\lim_{i \to \infty} a_i = 0 \qquad \sum_{i=1}^{\infty} a_i = \infty \qquad \sum_{i=1}^{\infty} a_i^2 < \infty$$

and $E_{\widehat{\theta}_i}[X]$ is an approximation of $E_\theta[X]$ based on $\theta_i$.

## 3. Solving the moment equations

Adapting the RM step for the SAOM:

The MoM equation is:

$$E_\theta[S] = s$$

Let

$$\overline{S}_i \approx E_\theta[S]$$

according to $\widehat{\theta}_i$

The value of $\theta$ is iteratively updated according to:

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i\left(\overline{S}_i - s\right)$$



## Updating the value of $\theta$

### Example

Let us consider the *"Teenage Friends and Lifestyle Study"* data set.

We model the network evolution according to the following parameter

$$\theta = (\lambda_1, \lambda_2, \beta_{out}, \beta_{rec}, \beta_{trans})$$

The MoM equations are:

$$\begin{cases} E_\theta\left[s_{\lambda_1}(X(t_1), X(t_0) | X(t_0) = x(t_0))\right] = 477 \\ E_\theta\left[s_{\lambda_2}(X(t_2), X(t_1) | X(t_1) = x(t_1))\right] = 437 \\ E_\theta\left[s_{out}\left(X(t_1) | X(t_0) = x(t_0)\right)\right] = 909 \\ E_\theta\left[s_{rec}\left(X(t_1) | X(t_0) = x(t_0)\right)\right] = 548 \\ E_\theta\left[s_{trans}\left(X(t_1) | X(t_0) = x(t_0)\right)\right] = 1146 \end{cases}$$



## Updating the value of $\theta$

### Example

- Guess $\theta_0 = (7.45, 6.83, -1.61, 0, 0)$
- Simulate the network evolution 1000 times according to $\widehat{\theta}_0$
- Approximation of the expected values

  $\overline{S}_{\lambda_1} = 605.745$ $\overline{S}_{\lambda_2} = 573.715$

  $\overline{S}_{\beta_{out}} = 1151.886$ $\overline{S}_{\beta_{rec}} = 141.406$ $\overline{S}_{\beta_{trans}} = 270.118$
- Approximation of the moment equation

  $\overline{S}_{\lambda_1} - 477 = 128.745$ $\overline{S}_{\lambda_2} - 437 = 136.715$

  $\overline{S}_{\beta_{out}} - 909 = 242.886$ $\overline{S}_{\beta_{rec}} - 548 = -406.594$ $\overline{S}_{\beta_{trans}} - 1146 = -875.882$



## Updating the value of $\theta$

### Example

- Guess $\theta_1 = (7.1, 6.75, -1.70, 1.20, 0.25)$
- Simulate the network evolution 1000 times according to $\widehat{\theta}_1$
- Approximation of the expected values

  $\overline{S}_{\lambda_1} = 549.787$ $\overline{S}_{\lambda_2} = 532.551$

  $\overline{S}_{\beta_{out}} = 1478.988$ $\overline{S}_{\beta_{rec}} = 517.450$ $\overline{S}_{\beta_{trans}} = 1062.537$
- Approximation of the moment equation

  $\overline{S}_{\lambda_1} - 477 = 72.787$ $\overline{S}_{\lambda_2} - 437 = 95.551$

  $\overline{S}_{\beta_{out}} - 909 = 569.988$ $\overline{S}_{\beta_{rec}} - 548 = -30.550$ $\overline{S}_{\beta_{trans}} - 1146 = -83.463$

## Updating the value of $\theta$

### Example

- Guess $\theta_2 = (7.10, 6.75, -2.20, 1.40, 0.35)$
- Simulate the network evolution 1000 times according to $\widehat{\theta}_2$
- Approximation of the expected values

  $\overline{S}_{\lambda_1} = 446.853$ $\quad$ $\overline{S}_{\lambda_2} = 437.166$

  $\overline{S}_{\beta_{out}} = 1025.729$ $\quad$ $\overline{S}_{\beta_{rec}} = 414.484$ $\quad$ $\overline{S}_{\beta_{trans}} = 698.734$
- Approximation of the moment equation

  $\overline{S}_{\lambda_1} - 477 = -30.147$ $\quad$ $\overline{S}_{\lambda_2} - 437 = 0.166$

  $\overline{S}_{\beta_{out}} - 909 = 116.729$ $\quad$ $\overline{S}_{\beta_{rec}} - 548 = -133.516$ $\quad$ $\overline{S}_{\beta_{trans}} - 1146 = -447.266$

and so on...



## Updating the value of $\theta$

### Example

- Guess $\theta_i = (10.71, 8.79, -2.63, 2.16, 0.46)$
- Simulate the network evolution 1000 times according to $\widehat{\theta}_i$
- Approximation of the expected values

  $\overline{S}_{\lambda_1} = 476.022$ $\quad$ $\overline{S}_{\lambda_2} = 436.983$

  $\overline{S}_{\beta_{out}} = 906.809$ $\quad$ $\overline{S}_{\beta_{rec}} = 545.578$ $\quad$ $\overline{S}_{\beta_{trans}} = 1147.795$
- Approximation of the moment equation

  $\overline{S}_{\lambda_1} - 477 = -0.978$ $\quad$ $\overline{S}_{\lambda_2} - 437 = -0.017$

  $\overline{S}_{\beta_{out}} - 909 = -2.191$ $\quad$ $\overline{S}_{\beta_{rec}} - 548 = -2.422$ $\quad$ $\overline{S}_{\beta_{trans}} - 1146 = 1.795$



## 3.Solving the moment equations

### The Robbins-Monro (RM) algorithm - remarks

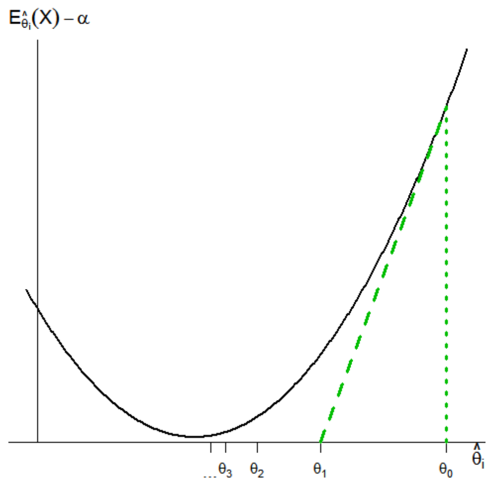
1. Convergence:
$$\lim_{i\to\infty} \widehat{\theta}_i = \theta$$
2. Modified RM step: to improve the convergence of the algorithm
$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1}\left(E_{\widehat{\theta}_i}[X] - \alpha\right)$$
   where $D$ is a diagonal matrix with elements
$$D = \frac{\partial}{\partial \widehat{\theta}_i} E_{\widehat{\theta}_i}[X]$$
   and estimate $\theta$ with:
$$\widehat{\theta} = \frac{1}{I}\sum_{i_1}^{I} \widehat{\theta}_i, \qquad I \; number \; of \; steps$$



## Updating the value of $\theta$

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1}\left(E_{\widehat{\theta}_i}[X] - \alpha\right)$$

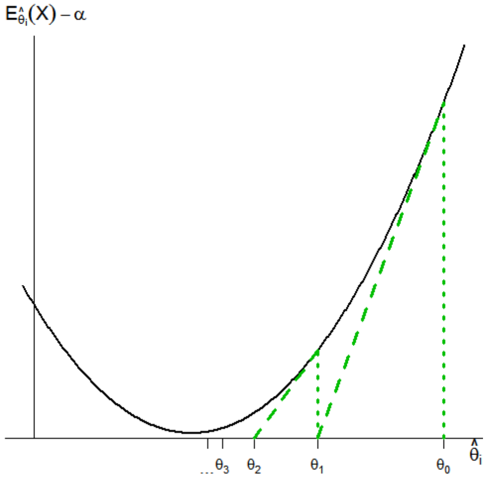
Intuitively:



## Updating the value of $\theta$

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1}\left(E_{\widehat{\theta}_i}[X] - \alpha\right)$$

Intuitively:

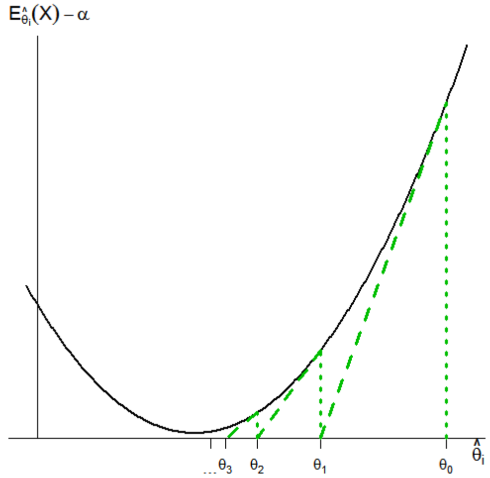





## Updating the value of $\theta$

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1}\left(E_{\widehat{\theta}_i}[X] - \alpha\right)$$

Intuitively:

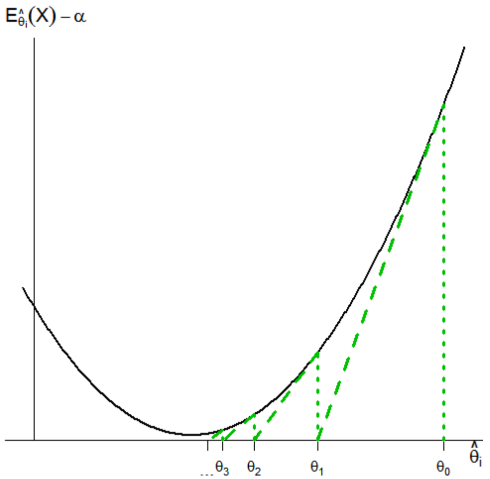





## Updating the value of $\theta$

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1}\left(E_{\widehat{\theta}_i}[X] - \alpha\right)$$

Intuitively:






## 3. Solving the moment equations

We want to solve:

$$E_\theta[S - s] = 0$$

using the RM step:

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1}\left(E_{\theta_i}[S] - s\right)$$

but we cannot write $\left(E_{\theta_i}[S] - s\right)$ and $D$ in a close form...

Approximate unknown quantities via Monte Carlo (MC) methods
$\Rightarrow$ **stochastic**

## Background: Monte Carlo method

Let $X$ be a random variable with distribution function $f_X(x)$.
We want to estimate the expected value $E[s(X)]$.

### Definition

The Monte Carlo method consists in:

1. generating a sample $(x_1, \cdots, x_q)$ from the distribution function $f_X(x)$
2. computing $s(x_l)$, $l = 1, \ldots, q$
3. approximating the expected value with the empirical average, i.e.:

$$\overline{S} = \frac{1}{q}\sum_{l=1}^{q} s(x_l)$$

### Reason

It can be proved that

$$\overline{S} \to E[s(X)]$$

as $q \to \infty$



## Background: Monte Carlo method - empirical proof

### Example

Let us consider the $\mathcal{G}(30, 0.08)$ model.

We are interested in estimating the expected number of edges.

Simulations from the $\mathcal{G}(30, 0.08)$ model for different values of $q$:

| q | 10 | 50 | 100 | 1000 | 10000 |
|---|---|---|---|---|---|
| $\overline{S}$ | 32.90 | 36.18 | 35.17 | 34.60 | 34.81 |

Since the number of edges $Y$ follows the binomial distribution,

$$E[Y] = Np = 34.8$$

Our results show that

$$\overline{S} \to E[Y]$$

as $q \to \infty$



## The MC Method for the statistics in the SAOM

1. Given $x(t_0)$ and $\theta$

$$x^{(1)}(t_1),\ x^{(1)}(t_2),\ \ldots,\ x^{(1)}(t_M)$$

$$\ldots$$

$$x^{(q)}(t_1),\ x^{(q)}(t_2),\ \ldots,\ x^{(q)}(t_M)$$

2. For each sequence compute the value $S^{(l)}$ taken by $S$
3. Approximate the expected value by

$$\overline{S} = \frac{1}{q}\sum_{l=1}^{q} S^{(l)} \to E_\theta[S]$$



## The MC Method for the expected values of the statistics

### Example

Approximating $E_\theta[s_{out}(X(t_1)|X(t_0) = x(t_0))]$ for the *"Teenage Friends and Lifestyle Study"* data set

1. Given:
   - $x(t_0)$
   - $\theta = (\lambda_1 = 10.69, \lambda_2 = 8.82, \beta_{out} = -2.63, \beta_{rec} = 2.17, \beta_{trans} = 0.46)$

   simulate the network evolution $q = 1000$ times

$$x^{(1)}(t_1),\ x^{(1)}(t_2),\ \ldots,\ x^{(1)}(t_M)$$

$$\ldots$$

$$x^{(q)}(t_1),\ x^{(q)}(t_2),\ \ldots,\ x^{(q)}(t_M)$$

```
myeff$initialValue[myeff$include] <- c(10.69,8.82,-2.63,2.17,0.46)
sim_model <- sienaModelCreate(projname = 'sim_model', cond = FALSE,
                              useStdInits = FALSE, nsub = 0 , n3=1000)
sim_ans <- siena07(sim_model, data = mydata, effects = myeff,returnDeps=TRUE)
```

## The MC Method for the expected values of the statistics

### Example

2. Compute the value assumed by $S_{out}$ for each sequence of networks

$$S_{out}^{(l)} = \sum_{m=1}^{M-1} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^{(l)}(t_m)$$

| sim | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|
| Nr. Edges | 942 | 874 | 1047 | 881 | 865 | 866 | 999 | 948 | ... |

```
stats <- t(t(sim_ans$sf) + sim_ans$targets)
stats
```

```
     [,1] [,2] [,3] [,4] [,5]
[1,]  480  447  942  570 1267
[2,]  469  450  874  524 1213
[3,]  514  458 1047  632 1567
[4,]  475  441  881  532 1118
[5,]  470  434  865  490  771
[6,]  457  422  866  510 1014
[7,]  509  469  999  632 2078
[8,]  483  428  948  564 1357
[9,]  525  474  962  586 1676
```



## The MC Method for the expected values of the statistics

### Example

3. Approximate the expected value by

$$\overline{S}_{out} = \frac{1}{q} \sum_{i=1}^{q} S_{out}^{(l)}$$

$$\overline{S}_{out} = \frac{942 + 874 + 1047 + 881 + 865 + 866 + 999 + 948 + \ldots}{1000} \approx 912$$

```
statsMC <- apply(stats,2,mean)
statsMC
```

```
475.492  438.963  911.737  550.602  1155.026
```



## 3. Solving the moment equations

We want to solve:

$$E_\theta[S - s] = 0$$

and using the RM:

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1} \left( E_{\theta_i}[S] - s \right)$$

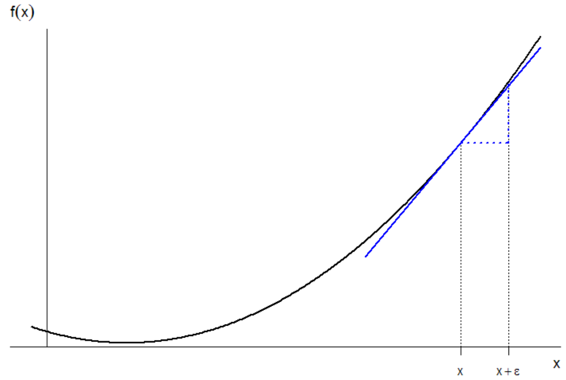
but we cannot write $\left( E_{\theta_i}[S] - s \right)$ and $D$ in a close form...

Approximate unknown quantities via Monte Carlo (MC) methods
$\Rightarrow$ **stochastic**



## Background: the MC Method for the approximation of derivatives

*Univariate function:*

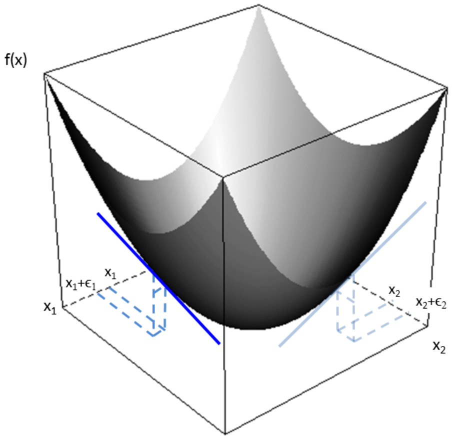



$$\frac{\partial}{\partial x} f(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

(Finite difference method)

## Background: the MC Method for the approximation of derivatives

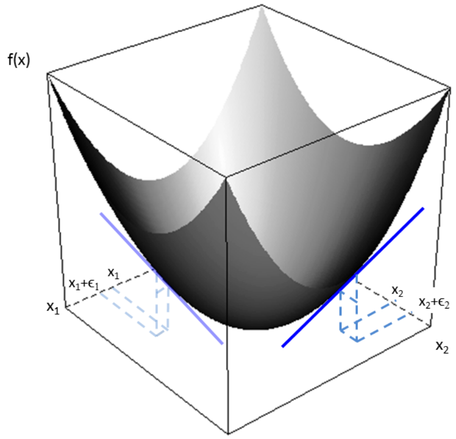
*Multivariate function* $x = (x_1, \ldots, x_J)$*:*






## Background: the MC Method for the approximation of derivatives

*Multivariate function* $x = (x_1, \ldots, x_J)$*:*



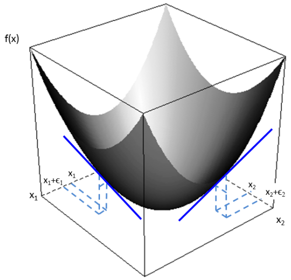



## The MC Method for the derivatives of the statistics

*Multivariate function* $x = (x_1, \ldots, x_J)$*:*

$J$ derivatives, one with respect to each variable

$$D = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) & & & \\ & \frac{\partial}{\partial x_2} f(x) & & \\ & & \ddots & \\ & & & \frac{\partial}{\partial x_J} f(x) \end{bmatrix}$$



The computation is done incrementing each variable at a time:

$e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ $j$-th unit vector

$\epsilon = (\epsilon_1, \ldots, \epsilon_J)$ vector of increments

$$\frac{\partial}{\partial x_j} f(x) = \lim_{\epsilon_j \to 0} \frac{f(x + \epsilon e_j) - f(x)}{\epsilon_j}$$



## The MC Method for the derivatives of the statistics

For the SAOM $D$ is a $M - 1 + K$ squared diagonal matrix:

$$D = \begin{bmatrix} \frac{\partial}{\partial \lambda_1} E_\theta[S-s] & & & & \\ & \ddots & & & \\ & & \frac{\partial}{\partial \lambda_{M-1}} E_\theta[S-s] & & & \\ & & & \frac{\partial}{\partial \beta_1} E_\theta[S-s] & & \\ & & & & \ddots & \\ & & & & & \frac{\partial}{\partial \beta_K} E_\theta[S-s] \end{bmatrix}$$

The diagonal element are computed increasing one parameter at a time by a "small" value $\epsilon_j$

$$\frac{\partial}{\partial \theta_j} E[S-s] = \lim_{\epsilon_j \to 0} \frac{E_{\theta+\epsilon e_j}[S-s] - E_\theta[S-s]}{\epsilon_j} \approx \frac{\overline{S}_{\theta+\epsilon e_j} - \overline{S}_\theta}{\epsilon_j}$$

## The MC Method for the expected values of the statistics

### Example

Approximating $\frac{\partial}{\partial \beta_{out}} E_\theta[s_{out}(X)]$ for the *"Teenage Friends and Lifestyle Study"* data set, considering a model with outdegree, reciprocity and transitivity

1. Given:
   - $x(t_0)$
   - $\theta = (10.69, 8.82, -2.63, 2.17, 0.46)$

   simulate the network evolution $q = 1000$ times w.r.t. $\theta$

$$x^{(1)}(t_1),\ x^{(1)}(t_2),\ \ldots,\ x^{(1)}(t_M)$$
$$\ldots$$
$$x^{(q)}(t_1),\ x^{(q)}(t_2),\ \ldots,\ x^{(q)}(t_M)$$

```
myeff$initialValue[myeff$include] <- c(10.69,8.82,-2.63,2.17,0.46)
sim_model <- sienaModelCreate(projname = 'sim_model', cond = FALSE,
                              useStdInits = FALSE, nsub = 0 , n3=1000)
sim_ans <- siena07(sim_model, data = mydata, effects = myeff,returnDeps=TRUE)
```



## The MC Method for the expected values of the statistics

### Example

2. Given:
   - $e_3 = (0, 0, 1, 0, 0)$ unit vector
   - $\epsilon = (0, 0, 0.1, 0, 0)$ vector of increments
   - $\theta + \epsilon e_3 = (10.69, 8.82, -2.53, 2.17, 0.46)$

   simulate the network evolution $q = 1000$ times w.r.t. $\theta + \epsilon e_3$

$$x^{*(1)}(t_1),\ x^{*(1)}(t_2),\ \ldots,\ x^{*(1)}(t_M)$$
$$\ldots$$
$$x^{*(q)}(t_1),\ x^{*(q)}(t_2),\ \ldots,\ x^{*(q)}(t_M)$$

```
myeff$initialValue[myeff$include] <- c(10.69,8.82,-2.63,2.17,0.46)+c(0,0,0.1,0,0)
sim_model2 <- sienaModelCreate(projname = 'sim_model', cond = FALSE,
                               useStdInits = FALSE, nsub = 0 , n3=1000)
sim_ans2 <- siena07(sim_model, data = mydata, effects = myeff,returnDeps=TRUE)
```



## The MC Method for the expected values of the statistics

### Example

3. Approximate the expected value using the MC method

$$\overline{S}_{out} = \frac{1}{q}\sum_{i=1}^{q} S_{out}^{(l)} = 911.737 \qquad \overline{S}^{*}_{out} = \frac{1}{q}\sum_{i=1}^{q} S_{out}^{*(l)} = 928.749$$

4. Approximate the first order derivative by

$$\frac{\partial}{\partial_{\beta_{out}}} E[S-s] = \lim_{\epsilon_3 \to 0} \frac{E_{\theta+\epsilon e_3}[S-s] - E_\theta[S-s]}{\epsilon_{\lambda_1}} \approx$$

$$\approx \frac{\overline{S}_{\theta+\epsilon e_3} - \overline{S}_\theta}{\epsilon_3} = \frac{928.749 - 911.737}{0.1} = 1701.2$$

```
statsMC <- apply(t(t(sim_ans$sf) + sim_ans$targets),2,mean)
                                        475.492    438.963    911.737    550.602    1155.026
statsMC2 <- apply(t(t(sim_ans2$sf) + sim_ans2$targets),2,mean)
                                        478.205    439.442    928.749    562.564    1202.127
deriv <- (statsMC2[3]-statsMC[3])/0.01
                                                      1701.2
```



## The Robbins-Monro algorithm

**Phase 1:**
- estimation of $D$
- first update of $\theta$

**Phase 2:**
- estimation of $\theta$ through the RM step

**Phase 3:**
- estimation of the standard error of $\theta$
- checking the convergence of the algorithm

## The Robbins-Monro algorithm - Phase1

**Algorithm 2:** Robbins-Monro algorithm - Phase 1

**Input**: $\theta_0$, $s$, $q_1$,$\epsilon$
**Output**: $\widehat{\theta}_{q_1}$, $\widehat{D}$

$i \leftarrow 0$; $d \leftarrow 0$; $S_0 \leftarrow 0$
**while** $i < q_1$ **do**
  $i \leftarrow i+1$
  $S_{i0} \sim \theta_0$
  $S_0 \leftarrow S_0 + S_{i0}$
  **for** *j=1, ... (M+K-1)* **do**
    $S_{ij} \sim \theta_1 + \epsilon e_j$
    $d_{ij} \leftarrow \epsilon_j^{-1}(S_{ij} - S_{i0})$
    $d \leftarrow d + d_{ij} e_j$
$\overline{S} \leftarrow \frac{1}{q_1} S_0$
$\widehat{d} \leftarrow \frac{1}{q_1} d$
$\widehat{D} \leftarrow diag(\widehat{d})$
$\widehat{\theta}_{q_1} \leftarrow \theta_0 - \widehat{D}^{-1}(\overline{S} - s)$
**return** $\widehat{\theta}_{q_1}$, $\widehat{D}$



## The Robbins-Monro algorithm - Phase 2

**Algorithm 3:** Robbins-Monro algorithm - Phase2

**Input**: $\widehat{\theta}_{q_1}$, $\widehat{D}$, $s$
**Output**: $\widehat{\theta}$

$h \leftarrow 1$
$\widehat{\theta}_1 \leftarrow \widehat{\theta}_{q_1}$
**while** $h \leq c$ **do**
  $i \leftarrow 0$
  $\theta_0 \leftarrow \widehat{\theta}_h$
  **if** $i >= q_h^+$ *OR* $(i > q_h^-$ *AND* $(S_{ih} - s)(S_{(i-1)h} - s) < 0)$ **then**
    $i \leftarrow i+1$
    $S_i \sim \widehat{\theta}_i$
    $\widehat{\theta}_{i+1} \leftarrow \widehat{\theta}_i - a_h \widehat{D}^{-1}(\overline{S}_i - s)$
    $\theta \leftarrow \widehat{\theta}_{i+1} + \theta$
  $\widehat{\theta}_h \leftarrow \frac{1}{i}\theta$
  $a_{h+1} \leftarrow a_h/2$
  $h \leftarrow h+1$;
$\widehat{\theta} \leftarrow \theta_c$
**return** $\widehat{\theta}$



## The Robbins-Monro algorithm - Phase 3

**Algorithm 4:** Robbins-Monro algorithm - Phase 3

**Input**: $\widehat{\theta}$, $s$, $q_3$,$\epsilon$
**Output**: $\widehat{\Sigma}_\theta$

$i \leftarrow 0$; $d \leftarrow 0$; $S_0 \leftarrow 0$
**while** $i < q_3$ **do**
  $i \leftarrow i+1$
  $S_{i0} \sim \widehat{\theta}$
  $S_0 \leftarrow S_0 + S_{i0}$
  **for** *j=1, ... M+K-1* **do**
    $S_{ij} \sim \widehat{\theta} + \epsilon e_j$
    $d_{ij} \leftarrow \epsilon_j^{-1}(S_{ij} - S_{i0})$
    $d \leftarrow d + d_{ij} e_j$
$\overline{S} = \frac{1}{q_3} S_0$
$\widehat{d} = \frac{1}{q_3} d$
$\widehat{D} = diag(D)$
$\widehat{\Sigma}_\theta = \widehat{D}^{-1} \left[ \frac{1}{q_3}(S_{i0} - \overline{S})(S_{i0} - \overline{S}) \right] \widehat{D}^{-1}$
**return** $\widehat{\Sigma}_\theta$



## Recap: estimating the parameter of the SAOM

### Issue

Given

$$x(t_0),\ x(t_1), \ldots,\ x(t_M)$$

and a parametrization of the SAOM

$$\theta = (\lambda_1,\ \ldots, \lambda_{M-1}, \beta_1,\ \ldots, \beta_K)$$

we want to estimate $\theta$ in a plausible way.

Different estimation methods are available:

1. Method of Moments:
   an estimation for $\theta$ is the value $\widehat{\theta}$ that solves:

$$E_\theta[S - s] = 0$$

## Recap: estimating the parameter of the SAOM

Given an initial guess $\theta_0$ for the parameter $\theta$, the procedure can be roughly depicted as follows:

$$\begin{array}{ccccc}
\theta_0 & \xrightarrow{simulation} & E_{\theta_0}[S-s] & \xrightarrow{rule} & \theta_1 \\
\theta_1 & \xrightarrow{simulation} & E_{\theta_1}[S-s] & \xrightarrow{rule} & \theta_2 \\
\dots & \xrightarrow{simulation} & \dots & \xrightarrow{rule} & \dots \\
\theta_{i-1} & \xrightarrow{simulation} & E_{\theta_{i-1}}[S-s] & \xrightarrow{rule} & \theta_i \\
\dots & \xrightarrow{simulation} & \dots & \xrightarrow{rule} & \dots
\end{array}$$

until a certain criterion is satisfied



## Recap: estimating the parameter of the SAOM

### Issue

Given

$$x(t_0),\ x(t_1), \dots,\ x(t_M)$$

and a parametrization of the SAOM

$$\theta = (\lambda_1,\ \dots, \lambda_{M-1}, \beta_1,\ \dots, \beta_K)$$

we want to estimate $\theta$ in a plausible way.

Different estimation methods are available:

1. Method of Moments:
   an estimation for $\theta$ is the value $\widehat{\theta}$ that solves:
   $$E_\theta[S-s] = 0$$
2. Maximum Likelihood Estimation:
   what is the most likely value of $\theta$ that could have generated the observed data?



## The Maximum-likelihood estimation (MLE)

### Definition

Let

- $$\mathcal{F} = \{F(\theta), \theta \in \Theta \subseteq \mathbb{R}^k\}$$
  be a collection of SAOMs parametrized by $\theta \in \Theta \subseteq \mathbb{R}^k$
- $x(t_0), \dots, x(t_M)$ be the observed data

The likelihood function associated with the observed data is:

$$L : \Theta \to \mathbb{R}; \theta \longmapsto P_\theta(x(t_0), \dots, x(t_M))$$

A parameter vector $\widehat{\theta}$ maximizing $L$:

$$\widehat{\theta} = arg \max_{\theta \in \Theta} L(\theta)$$

is called a *maximum likelihood estimate* for $\theta$



## Computing the Likelihood function

For semplicity let us consider only two observations $x(t_0)$ and $x(t_1)$

The model assumptions allow to decompose the process in a series of micro-steps:

$$\{(T_r, i_r, j_r), r = 1,\ \dots, R\}$$

where

- $T_r$ is the time point for an opportunity for change
- $i_r$ denotes the actor who has the opportunity to change
- $j_r$ is the actor towards whom the tie is changed

Let $R$ be the total number of micro-steps between $t_0$ and $t_1$.

We assume that the time point $T_r$ are ordered increasingly:

$$t_0 = T_0 < T_1 < \dots < T_R < t_1$$

## Computing the Likelihood function

### Definition

Given the sequence $\{(T_r, i_r, j_r), r = 1, \ldots, R\}$

$$L(\theta) = \prod_{r=1}^{R} P_\theta((T_r, i_r, j_r)) \propto \frac{(n\lambda)^R}{R!} e^{-n\lambda} \prod_{r=1}^{R} \frac{1}{n} p_{i_r j_r}(\beta, x(T_r))$$

Then, the estimate for $\theta$ is

$$\widehat{\theta} = arg \max_{\theta \in \Theta} L(\theta)$$

or equivalently

$$\widehat{\theta} = arg \max_{\theta \in \Theta} log(L(\theta))$$

where $log(L(\theta))$ is called *the log-likelihood function*



## The Maximum-likelihood estimation (MLE)

In practice finding

$$\widehat{\theta} = arg \max_{\theta \in \Theta} log(L(\theta))$$

means determining $\widehat{\theta}$ such that:

$$\frac{\partial}{\partial \theta} log(L(\theta)) = 0$$

where $\frac{\partial}{\partial \theta} log(L(\theta))$ is called *score function*.



## The Maximum-likelihood estimation (MLE)

### Example

Let us consider the *"Teenage Friends and Lifestyle Study"* data set.

We model the network evolution according to the following parameter

$$\theta = (\lambda_1, \lambda_2, \beta_{out}, \beta_{rec}, \beta_{trans})$$

We look for $\widehat{\theta}$ such that:

$$\begin{cases} \frac{\partial}{\partial \lambda_1} log(L(\theta)) = 0 \\ \frac{\partial}{\partial \lambda_2} log(L(\theta)) = 0 \\ \frac{\partial}{\partial \beta_{out}} log(L(\theta)) = 0 \\ \frac{\partial}{\partial \beta_{rec}} log(L(\theta)) = 0 \\ \frac{\partial}{\partial \beta_{trans}} log(L(\theta)) = 0 \end{cases}$$



## The Maximum-likelihood estimation (MLE)

Problem:
we cannot observe the complete data, i.e., the complete series of micro-steps that lead from $x(t_0)$ to $x(t_1)$, from $x(t_1)$ to $x(t_2)$, ...

$\Downarrow$

we cannot compute the $L$ of the observed data

$\Downarrow$

a stochastic approximation method must be applied.

## The Maximum-likelihood estimation (MLE)

Given an initial guess $\theta_0$ for the parameter $\theta$, the procedure can be roughly depicted as follows:

$$\begin{array}{ccccc}
\theta_0 & \xrightarrow{simulation} & \frac{\partial}{\partial\theta} log(L(\theta_0)) & \xrightarrow{rule} & \theta_1 \\
\theta_1 & \xrightarrow{simulation} & \frac{\partial}{\partial\theta} log(L(\theta_1)) & \xrightarrow{rule} & \theta_2 \\
\ldots & \xrightarrow{simulation} & \ldots & \xrightarrow{rule} & \ldots \\
\theta_{i-1} & \xrightarrow{simulation} & \frac{\partial}{\partial\theta} log(L(\theta_{i-1})) & \xrightarrow{rule} & \theta_i \\
\ldots & \xrightarrow{simulation} & \ldots & \xrightarrow{rule} & \ldots
\end{array}$$

until a certain criterion is satisfied



## Augmented data

To approximate the likelihood we use the *augmented data method*

### Definition

The *augmented data* (or *sample path*) consist of the sequence of tie changes that brings the network from $x(t_0)$ to $x(t_1)$

$$(i_1,j_1),\ldots,(i_R,j_R)$$

Formally:

$$\underline{v} = \{(i_1,j_1),\ldots,(i_R,j_R)\} \in \mathcal{V}$$

where $\mathcal{V}$ is the set of all sample paths connecting $x(t_0)$ and $x(t_1)$.

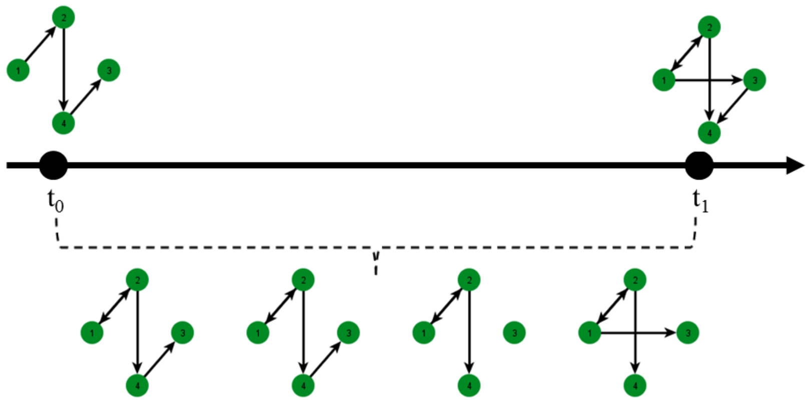
We can approximate the likelihood function (and then the score function) of the observed data using the probability of $\underline{v}$

$$P(\underline{v}|x(t_0),x(t_1)) \propto \frac{(n\lambda)^R}{R!} e^{-n\lambda} \prod_{r=1}^{R} \frac{1}{n} p_{i_r j_r}(\beta, x(T_r))$$



## Approximating the Likelihood function

### Example




$$\underline{v} = \{(2,1),(1,1),(4,3),(1,3),(3,4)\}$$



## Augmented data

### Example


$$\mathcal{V} = \{\{(2,1),(1,1),(4,3),(1,3),(3,4)\},$$

$$\{(1,2),(3,4),(1,3),(4,3),(2,1),(1,2)\},$$

$$\{(3,3),(4,4),(2,3),(4,3),(2,1),(2,3),(3,4),(1,1)\},$$

$$\ldots\}$$

## Sampling the augmented data

How to sample the augmented data from the distribution:

$$P(\underline{v}|x(t_0),x(t_1)) \propto \frac{(n\lambda)^R}{R!} e^{-n\lambda} \prod_{r=1}^{R} \frac{1}{n} p_{i_r j_r}(\beta, x(T_r))$$

given a certain value of the parameter $\theta$?

The augmented data are sampled through the *Metropolis-Hastings algorithm*



## Sampling the augmented data

The *Metropolis-Hastings algorithm* is defined by the following steps:

1. given $\underline{v}_i = \underline{v}$, generate $\widetilde{\underline{v}}$ from a proposal distribution $u(\widetilde{\underline{v}}|\underline{v}_i)$

   The proposal distribution $u(\widetilde{\underline{v}}|\underline{v}_i) : \mathcal{V} \to [0,1]$ assigns non-zero probabilities only to the following 5 cases:

   a. *Pairwise deletions*

   b. *Pairwise insertions*

   c. *Single deletion*

   d. *Single insertion*

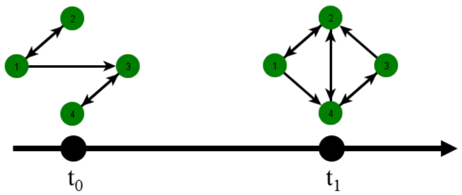
   e. *Permutation*



## Sampling the augmented data

a. *Pairwise deletions*: $r_1$ and $r_2$ such that $(i_{r_1}, j_{r_1}) = (i_{r_2}, j_{r_2})$ is selected and the pairs $(i_{r_1}, j_{r_1})$ and $(i_{r_2}, j_{r_2})$ are deleted from $\underline{v}$

### Example




$\underline{v} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$

- Select at random $(r_1, r_2)$ in $\{(1,7),(1,10),(7,10),(2,8)\}$, e.g. $(r_1, r_2) = (1,7)$
- Delete the elements $(2,4)$

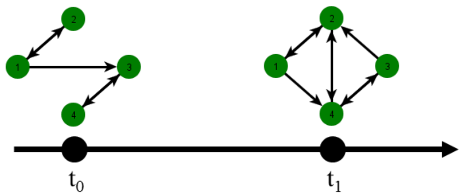
  $\widetilde{\underline{v}} = (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$



## Sampling the augmented data

b. *Pairwise insertions*: $(i,j) \in \mathcal{N}^2$ and $r_1$ and $r_2$ are randomly chosen. The element $(i,j)$ is inserted in $\underline{v}$ immediately before $r_1$ and $r_2$

### Example




$\underline{v} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$

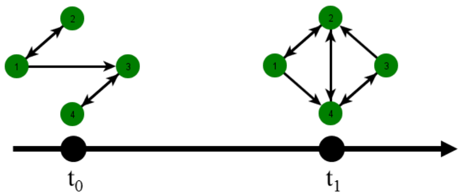
- Select at random $(i,j)$ and $(r_1, r_2)$, e.g. $i=4$, $j=1$, $r_1=5$, $r_2=7$
- Insert the elements $(4,1)$ before $r_1 = 5$ and $r_2 = 7$

$\widetilde{\underline{v}} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ \mathbf{(4,1)}\ (3,2)\ (1,4)\ \mathbf{(4,1)}\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$

## Sampling the augmented data

c. *Single deletion*: one pair $(i_r, j_r)$ satisfying $i_r = j_r$ is randomly selected and deleted from $\underline{v}$

### Example




$\underline{v} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$

- Select at random $r$ in $\{3,11\}$, e.g. $r = 11$
- Delete the elements $(3,3)$

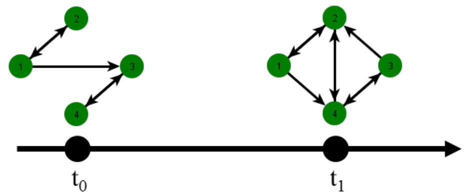
$\widetilde{\underline{v}} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ (2,4)\ (2,3)\ (1,3)\ (2,4)$



## Sampling the augmented data

d. *Single insertion*: one actor $i \in \mathcal{N}$ and an index $r$ are selected. The element $(i,i)$ is inserted immediately before $r$

### Example




$\underline{v} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$

- Select at random $i \in \mathcal{N}$ and $r$, e.g. $i = 4$ $r = 6$
- Insert the elements $(4,4)$ before $r = 6$

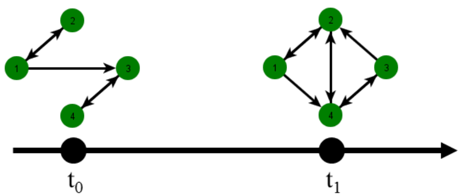
$\widetilde{\underline{v}} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ \mathbf{(4,4)}\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$



## Sampling the augmented data

e. *Permutations*: for randomly chosen indices $r_1 < r_2$, the sequence $(i_{r_1}, j_{r_1}), \ldots, ((i_{r_2}, j_{r_2}))$ is randomly permuted

### Example




$\underline{v} = (2,4)\ (2,3)\ (1,1)\ (4,2)\ (3,2)\ (1,4)\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$

- Select at random $(r_1, r_2)$ and $r$, e.g. $r_1 = 2, r_2 = 5$
- Permute the sequence $(i_2, j_2), \ldots, (i_5, j_5)$

$\underline{v} = (2,4)\ \mathbf{(1,1)}\ \mathbf{(2,3)}\ \mathbf{(3,2)}\ \mathbf{(4,2)}\ (1,4)\ (4,4)\ (2,4)\ (2,3)\ (1,3)\ (2,4)\ (3,3)$



## Sampling the augmented data

The *Metropolis-Hastings algorithm* is defined by the following steps:

1. given $\underline{v}_i = \underline{v}$, generate $\widetilde{\underline{v}}$ from the proposal distribution $u(\widetilde{\underline{v}} \,|\, \underline{v}_i)$

2. take

$$\underline{v}_{i+1} = \begin{cases} \widetilde{\underline{v}} & \textit{with probability} \quad \rho(\widetilde{\underline{v}}, \underline{v}) \\ \underline{v} & \textit{with probability} \quad 1 - \rho(\widetilde{\underline{v}}, \underline{v}) \end{cases}$$

where

$$\rho(\widetilde{\underline{v}}, \underline{v}) = \min\left\{ \frac{P(\widetilde{\underline{v}})u(\underline{v} \,|\, \widetilde{\underline{v}})}{P(\underline{v})u(\widetilde{\underline{v}} \,|\, \underline{v})}, 1 \right\}$$

The transition probabilities of the chain generate by the Metropolis-Hastings algorithm are given by $\rho(\widetilde{\underline{v}}, \underline{v})u(\widetilde{\underline{v}} \,|\, \underline{v})$

## Sampling the augmented data

### Theorem

*The Metropolis-Hastings algorithm leads to an irreducible, aperiodic and reversible Markov-chain with stationary distribution:*

$$P(\underline{v}|x(t_0),x(t_1)) \propto \frac{(n\lambda)^R}{R!} e^{-n\lambda} \prod_{r=1}^{R} \frac{1}{n} p_{i_r j_r}(\beta, x(T_r))$$

### Proof

- *The Markov chain is irreducible.*
  Pairwise deletions and insertions and single deletion and insertion are sufficient for all $\underline{v} \in \mathcal{V}$ to communicate.
- *The Markov chain is aperiodic.*
  The graph associated to the resulting Markov-chain contains all the loops and thus the greatest common divisor of all cycles is one.



## Sampling the augmented data

- *The Markov chain is reversible.* The detailed balance condition:

$$\rho(\widetilde{\underline{v}},\underline{v})u(\widetilde{\underline{v}}|\underline{v})P(\underline{v}) = \rho(\underline{v},\widetilde{\underline{v}})u(\underline{v}|\widetilde{\underline{v}})P(\widetilde{\underline{v}})$$

is satisfied.

$$\rho(\widetilde{\underline{v}},\underline{v})u(\widetilde{\underline{v}}|\underline{v})P(\underline{v}) = \min\left\{\frac{P(\widetilde{\underline{v}})u(\underline{v}|\widetilde{\underline{v}})}{P(\underline{v})u(\widetilde{\underline{v}}|\underline{v})},1\right\} u(\widetilde{\underline{v}}|\underline{v})P(\underline{v}) =$$

$$= \min\left\{\frac{P(\widetilde{\underline{v}})u(\underline{v}|\widetilde{\underline{v}})}{u(\widetilde{\underline{v}}|\underline{v})}, P(\underline{v})\right\} u(\widetilde{\underline{v}}|\underline{v}) =$$

$$= \min\left\{\frac{u(\underline{v}|\widetilde{\underline{v}})}{u(\widetilde{\underline{v}}|\underline{v})}, \frac{P(\underline{v})}{P(\widetilde{\underline{v}})}\right\} u(\widetilde{\underline{v}}|\underline{v})P(\widetilde{\underline{v}}) =$$

$$= \min\left\{1, \frac{P(\underline{v})u(\widetilde{\underline{v}}|\underline{v})}{P(\widetilde{\underline{v}})u(\underline{v}|\widetilde{\underline{v}})}\right\} u(\underline{v}|\widetilde{\underline{v}})P(\widetilde{\underline{v}}) =$$

$$= \rho(\underline{v},\widetilde{\underline{v}})u(\underline{v}|\widetilde{\underline{v}})P(\widetilde{\underline{v}})$$



## Sampling the augmented data

The ML estimation algorithm can be sketched in the following way:

1. For each $m = 1,\ldots,M-1$ makes a large number of Metropolis-Hastings steps yielding $v^{(i)} = (v_1^{(i)},\ldots,v_{M-1}^{(i)})$
2. Compute the score function:
$$\frac{\partial}{\partial\theta} log(L(\widehat{\theta}_i;x;v_m^{(i)}))$$
3. Update the parameter estimate using the Robbins-Monro step
$$\theta_{i+1} = \theta_i + a_i D^{-1} U(L(\widehat{\theta}_i;x;v_m^{(i)}))$$

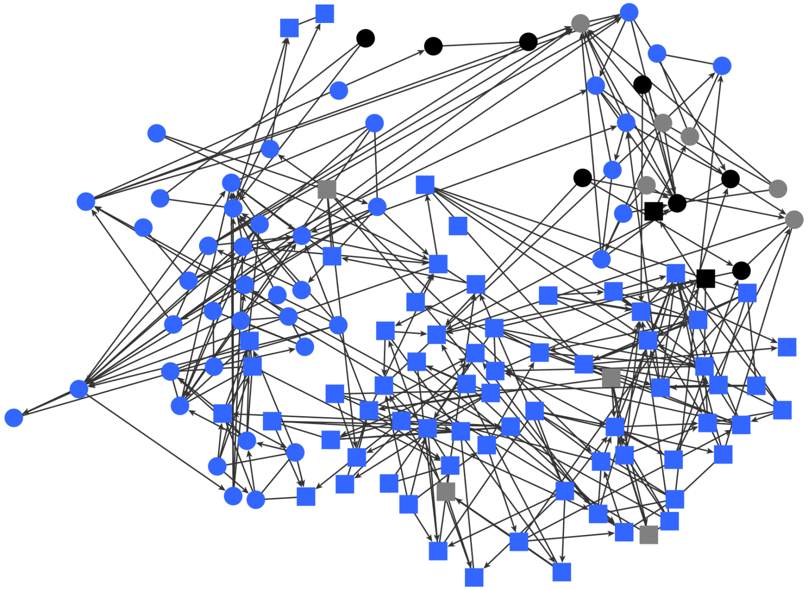
The estimate $\widehat{\theta}$ is calculated as the average of the $\theta_{i+1}$ values generated by this algorithm.



## Outline

## Networks are dynamic by nature: a real example

Ties and actors'characteristics can change over time.





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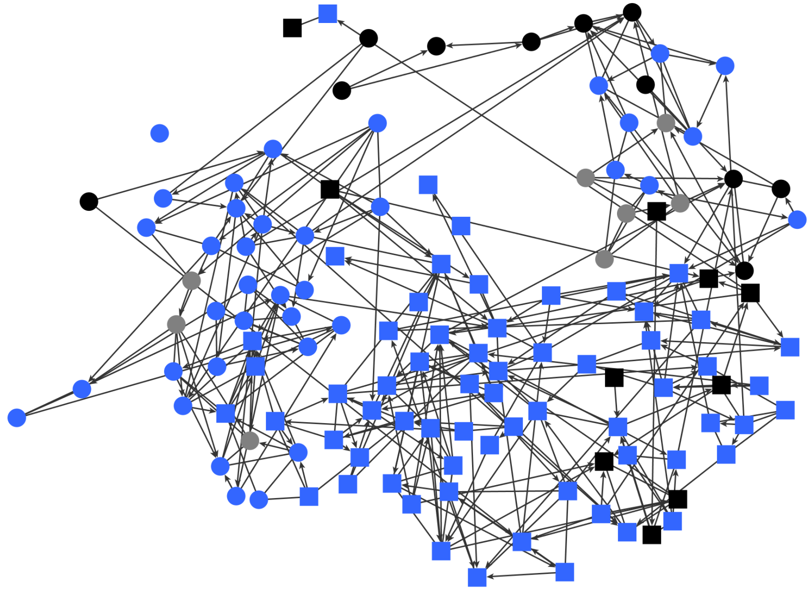



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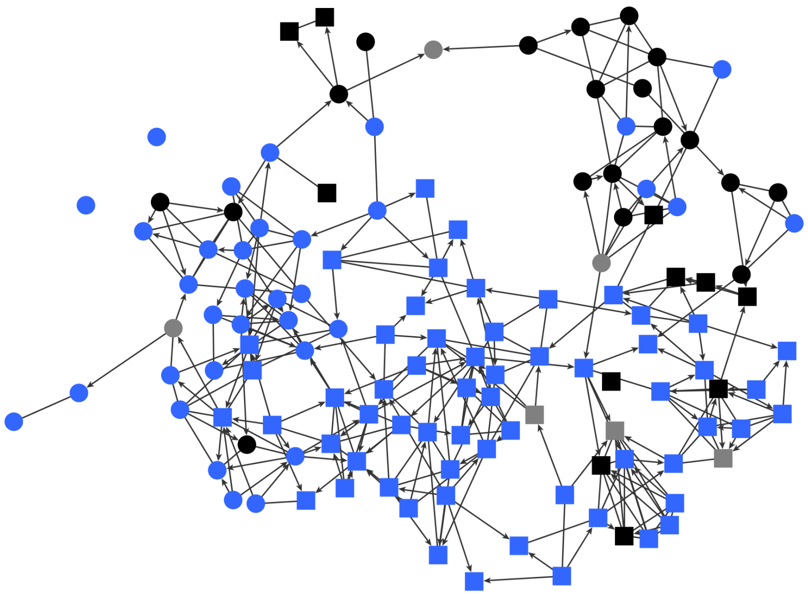




## Motivation

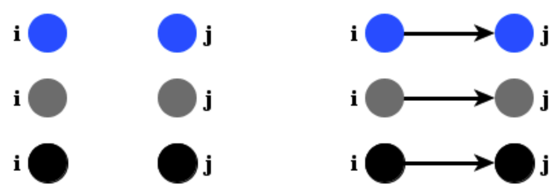
1. Social network dynamics can depend on actors'characteristics.

Selection process: relationship *partners* are selected according to their characteristics

Example

Homophily: the formation of relations based on the similarity of two actors

E.g. smoking behavior



## Motivation

2. Changeable actors'characteristics can depend on the social network

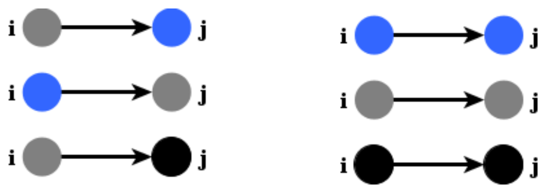
E.g.: opinions, attitudes, intentions, etc. - we use the word behavior for all of these!

Influence process: actors adjust their characteristics according to the characteristics of other actors to whom they are tied

Example

Assimilation/contagion: connected actors become increasingly similar over time

E.g. smoking behavior






## Competing explanatory stories

Homophily and assimilation give rise to the same outcome (similarity of connected individuals)

$$\Downarrow$$

study of influence requires the consideration of selection and vice versa.

Fundamental question: is this similarity caused mainly by influence or mainly by selection?

Extending the SAOM for the co-evolution of networks and behaviors



## Competing explanatory stories

Example

Similarity in smoking:

Selection: "a smoker may tend to have smoking friends because, once somebody is a smoker, he or she is likely to meet other smokers in smoking areas and thus has more opportunities to form friendship ties with them"

Influence: "the friendship with a smoker may have made an actor smoking in the first place"



## Longitudinal network-behavior panel data

1. a network $x$ represented by its adjacency matrix

2. a series of actors' attributes:
   - $H$ constant covariates $V_1, \cdots, V_H$
   - $L$ behavior covariates $Z_1(t), \cdots, Z_L(t)$
     Behavior variables are ordinal categorical variables.

Longitudinal network-behavior panel data: networks and behaviors observed at $M \geq 2$ time points $t_1, \cdots, t_M$

$$(x,z)(t_0),\ (x,z)(t_1), \cdots,\ (x,z)(t_M)$$

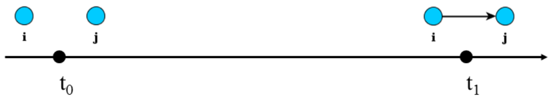
and the constant covariates $V_1, \cdots, V_H$.

## Assumptions

1. *Distribution of the process.*
   Changes between observational time points are modeled according to a continuous-time Markov chain.
   - *State space* $\mathcal{C}$*:* all the possible configurations arising from the combination of network and behaviors

     $$|C| = 2^{n(n-1)} \times B^n$$

     where $B$ is the number of categories for the behavior variable.
   - *Markovian assumption:* changes actors make are assumed to depend only on the current state of the network
   - *Continuous-time:*



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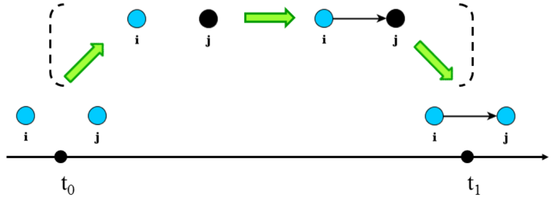




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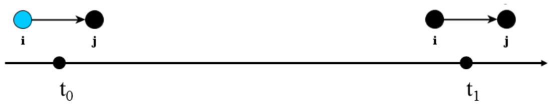




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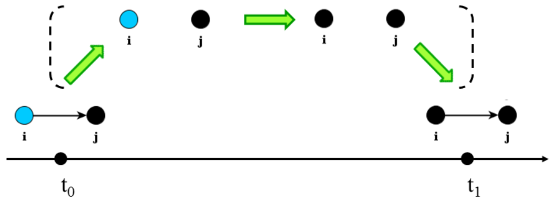
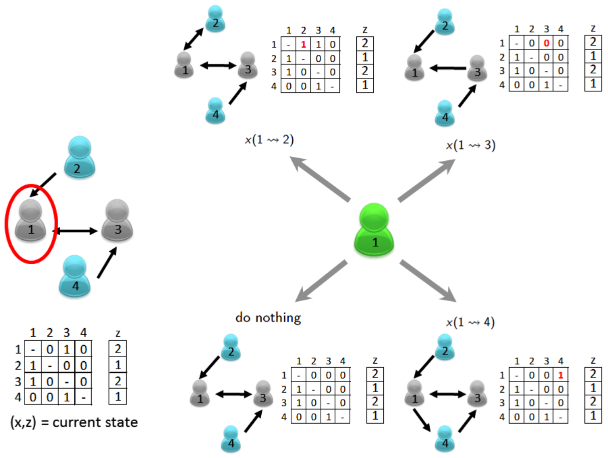


## Assumptions

2. *Opportunity to change.*
   At any given moment one probabilistically selected actor has the opportunity to change one of his outgoing ties or his behavior.



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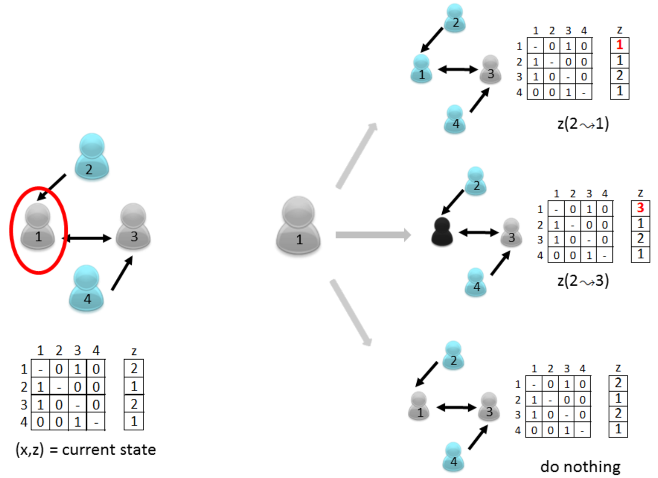
2. *Opportunity to change.*
   At any given moment one probabilistically selected actor has the opportunity to change one of his outgoing ties or his behavior.






## Assumptions

3. *Absence of co-occurrence.*
   At each instant $t$, only one actor has the opportunity to change (one of his outgoing ties or his behavior)

4. *Actor-oriented perspective.*
   Actors control their outgoing ties as well as their own behavior.

   - the actor decide to change one of his outgoing ties or his behavior trying to maximize *a utility function*
   - two distinct objective functions: one for the network and one for the behavior change
   - actors have complete knowledge about the network and the behaviors of all the the other actors
   - the maximization is based on immediate returns (myopic actors)



## Model definition

The co-evolution process is decomposed into a series of micro-steps:

- the opportunity of changing one network tie and the corresponding tie changed
- the opportunity of changing a behavior and the corresponding unit changed in behavior

$$\Downarrow$$

every micro-step requires the identification of a focal actor who gets the opportunity to make a change and the identification of the change outcome

| | Occurrence | *Preference* |
|---|---|---|
| Network changes | Network rate function | Network objective function |
| Behavioral changes | Behavioral rate function | Behavioral objective function |

## The rate functions

The frequency by which actors have the opportunity to make a change is modeled by the *rate functions*, one for each type of change.

Why must we specify two different rate functions?

Practically always, one type of decision will be made more frequently than the other

Example

In the joint study of friendship and smoking behavior at high school, we would expect more frequent changes in the network than in behavior



## The rate functions

**Network rate function**
$T_i^{net}$ = the waiting time until $i$ gets the opportunity to make a network change

$$T_i^{net} \sim Exp(\lambda_i^{net})$$

**Behavior rate function**
$T_i^{beh}$ = the waiting time until $i$ gets the opportunity to make a behavior change

$$T_i^{beh} \sim Exp(\lambda_i^{beh})$$

**Waiting time for a new micro-step**
$T_i^{net \vee beh}$ = the waiting time until $i$ gets the opportunity to make any change

$$T_i^{net \vee beh} \sim Exp(\lambda_{tot})$$

where

$$\lambda_{tot} = \sum_i (\lambda_i^{net} + \lambda_i^{beh})$$



## The rate functions (simplest specification)

**Network rate function**
$T_i^{net}$ = the waiting time until $i$ gets the opportunity to make a network change

$$T_i^{net} \sim Exp(\lambda^{net})$$

**Behavior rate function**
$T_i^{beh}$ = the waiting time until $i$ gets the opportunity to make a behavior change

$$T_i^{beh} \sim Exp(\lambda^{beh})$$

**Waiting time for a new micro-step**
$T_i^{net \vee beh}$ = the waiting time until $i$ gets the opportunity to make any change

$$T_i^{net \vee beh} \sim Exp(\lambda_{tot})$$

where

$$\lambda_{tot} = n(\lambda^{net} + \lambda^{beh})$$



## The rate functions (simplest specification)

**Probabilities for an actor to make a micro-step**

$$P(i\ can\ make\ a\ network\ micro-step) = \frac{\lambda^{net}}{\lambda_{tot}}$$

$$P(i\ can\ make\ a\ behavioral\ micro-step) = \frac{\lambda^{beh}}{\lambda_{tot}}$$

**Probabilities for a micro-step**

$$P(network\ micro-step) = \frac{n\lambda^{net}}{\lambda_{tot}} = \frac{\lambda^{net}}{\lambda^{net} + \lambda^{beh}}$$

$$P(behavioral\ micro-step) = \frac{n\lambda^{beh}}{\lambda_{tot}} = \frac{\lambda^{beh}}{\lambda^{net} + \lambda^{beh}}$$

## The objective functions

Why must we specify two different objective functions?

- The network objective function represents how likely it is for $i$ to change one of his outgoing ties
- The behavioral objective function represents how likely it is for the actor $i$ the current level of his behavior

**Network utility function**

$$u_i^{net}(\beta, x(i \rightsquigarrow j), z, v) = f_i^{net}(\beta, x(i \rightsquigarrow j), z, v) + \epsilon_i(t, x, j)$$
$$= \sum_{k=1}^{K} \beta_k s_{ik}^{net}(x, z, v) + \epsilon_i(t, x, j)$$



## The objective functions

**Behavioral utility function**

$$u_i^{beh}(\gamma, z(l \rightsquigarrow l'), x, v) = f_i^{beh}(\gamma, z(l \rightsquigarrow l'), x, v) + \epsilon_i(t, z, l, l')$$
$$= \sum_{w=1}^{W} \gamma_w s_{iw}^{beh}(x, z(l \rightsquigarrow l'), v) + \epsilon_i(t, z, l, l')$$

where

- $s_w^{beh}(x, z, v)$ are effects
- $\gamma_w$ are statistical parameters
- $\epsilon_i(t, z, l, l')$ is a random term

The probability that an actor $i$ changes his own behavior by one unit is:

$$p_{ll'}(\gamma, z(l \rightsquigarrow l'), x, v) = \frac{exp\left(f_i^{beh}(\gamma, z(l \rightsquigarrow l'), x, v)\right)}{\sum_{l'' \in \{l+1, l-1, l\}} exp\left(f_i^{beh}(\gamma, z(l \rightsquigarrow l''), x, v)\right)}$$

$p_{ll}$ is the probability that $i$ does not change his behavior



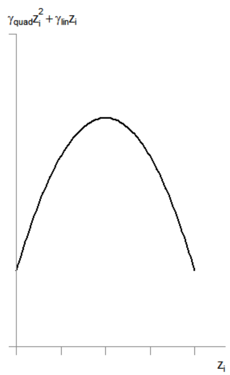
## The objective functions

**The specification of the behavioral objective function**

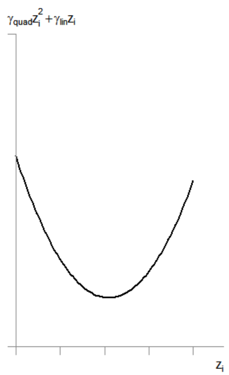
- *Basic shape effects*

$$s_{i\_linear}^{beh}(x, z, v) = z_i \qquad s_{i\_quadratic}^{beh}(x, z, v) = z_i^2$$

The basic shape effects must be always included in the model specification




negative $\gamma_{quad}$




positive $\gamma_{quad}$



## The objective functions

**The specification of the behavioral objective function**

- *Classical influence effects*

1. The *average similarity effect* $s_{i\_avsim}^{beh}(x, z, v)$

$$s_{i\_avsim}^{beh}(x, z, v) = \frac{1}{x_{i+}} \sum_{j=1}^{n} x_{ij}(sim_z(ij) - \widehat{sim}_z)$$

where

$$sim_z(ij) = 1 - \frac{|z_i - z_j|}{R_z}$$

$R_z$ is the range of the behavior $z$ and $\widehat{sim}_z$ is the mean similarity value

2. The *total similarity effect* $s_{i\_totsim}^{beh}(x, z, v)$

$$s_{i\_totsim}^{beh}(x, z, v) = \sum_{j=1}^{n} x_{ij}(sim_z(ij) - \widehat{sim}_z)$$

## The objective functions

**The specification of the behavioral objective function**

- *Position-dependent influence effects*

  Network position could also have an effect on the behavior of dynamics

  1. *outdegree effect*

  $$s^{beh}_{i\_out}(x,z,v) = z_i \sum_{j=1}^{n} x_{ij}$$

  2. *indegree effect*

  $$s^{beh}_{i\_ind}(x,z,v) = z_i \sum_{j=1}^{n} x_{ji}$$

- *Effects of other actor variables.*
  For each actor's attribute a main effect on the behavior can be included in the model



## Example

Example data: excerpt from the "Teenage Friends and Lifestyle Study" data set

We will use the SAOM for the co-evolution of networks and behaviors to distinguish influence from selection.

1. Do pupils select friends based on similar smoking behavior?

2. Are pupils influenced by friends to adjust to their smoking behavior?

Dependent variables: friendship networks and smoking behavior

Covariate: gender



## Precondition of the analysis

To find out whether it makes sense to analyze the data with a co-evolution model one should check whether:

1. the data are sufficiently informative to allow for identification of effects

$$J = \frac{N_{11}}{N_{11} + N_{01} + N_{10}} > 0.3 \qquad \textit{Jaccard index}$$

```
Tie changes between subsequent observations:
 periods        0 =>  0   0 =>  1   1 =>  0   1 =>  1   Distance Jaccard   Missing
  1 ==>  2      15827       237       240       208       477     0.304     0 (0%)
  2 ==>  3      15839       228       209       236       437     0.351     0 (0%)
```

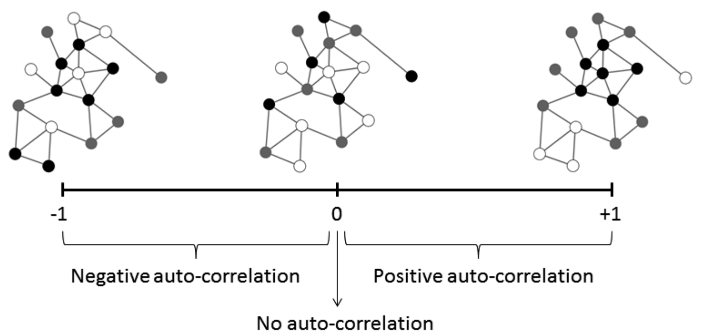


## Precondition of the analysis

2. there is interdependence between network and behavioral variables

$$I = \frac{n \sum_{ij} x_{ij}(z_i - \overline{z})(z_j - \overline{z})}{\left(\sum_{ij} x_{ij}\right)\left(\sum_{i} (z_i - \overline{z})^2\right)} \qquad \textit{Moran index}$$

where $\overline{z}$ is the mean of $z$ over all the periods



## Precondition of the analysis

The computation of the index $I$ for the data leads to

0.244 0.258 0.341

Conclusion:
there is considerable dependence between networks and behaviors and it is reasonable to apply the SAOM

```
moran1 <- nacf(net1,tobacco[,1],lag.max=1,neighborhood.type = "out",
           type="moran",mode="digraph")
moran2 <- nacf(net2,tobacco[,2],,lag.max=1,neighborhood.type = "out",
           type="moran",mode="digraph")
moran3 <- nacf(net3,tobacco[,3],,lag.max=1,neighborhood.type = "out",
           type="moran",mode="digraph")

moranInd <- c(moran1[2],moran2[2],moran3[2])
```



## Parameter interpretation: a baseline model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Network Dynamics* | | | |
| constant friendship rate (period 1) | 8.6287 | ( 0.6666 ) | |
| constant friendship rate (period 2) | 7.2489 | ( 0.5466 ) | |
| outdegree (density) | -2.4084 | ( 0.0407 ) | -59.1268 |
| reciprocity | 2.7024 | ( 0.0823 ) | 32.8337 |
| *Behavior Dynamics* | | | |
| rate smokebeh (period 1) | 3.8922 | ( 1.9689 ) | |
| rate smokebeh (period 2) | 4.4813 | ( 2.3679 ) | |
| behavior smokebeh linear shap | -3.5464 | ( 0.4394 ) | -8.0712 |
| behavior smokebeh quadratic shape | 2.8464 | ( 0.3628 ) | 7.8447 |

Network rate parameters:

- about 9 opportunities for a network change in the first period
- about 7 opportunities for a network change in the second period



## Parameter interpretation: a baseline model


| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Network Dynamics* | | | |
| constant friendship rate (period 1) | 8.6287 | ( 0.6666 ) | |
| constant friendship rate (period 2) | 7.2489 | ( 0.5466 ) | |
| outdegree (density) | -2.4084 | ( 0.0407 ) | -59.1268 |
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| *Behavior Dynamics* | | | |
| rate smokebeh (period 1) | 3.8922 | ( 1.9689 ) | |
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| behavior smokebeh linear shap | -3.5464 | ( 0.4394 ) | -8.0712 |
| behavior smokebeh quadratic shape | 2.8464 | ( 0.3628 ) | 7.8447 |


Network objective function parameters:

- outdegree parameter: the observed networks have low density
- reciprocity parameter: strong tendency towards reciprocated ties



## Parameter interpretation: a baseline model


| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Network Dynamics* | | | |
| constant friendship rate (period 1) | 8.6287 | ( 0.6666 ) | |
| constant friendship rate (period 2) | 7.2489 | ( 0.5466 ) | |
| outdegree (density) | -2.4084 | ( 0.0407 ) | -59.1268 |
| reciprocity | 2.7024 | ( 0.0823 ) | 32.8337 |
| *Behavior Dynamics* | | | |
| rate smokebeh (period 1) | 3.8922 | ( 1.9689 ) | |
| rate smokebeh (period 2) | 4.4813 | ( 2.3679 ) | |
| behavior smokebeh linear shap | -3.5464 | ( 0.4394 ) | -8.0712 |
| behavior smokebeh quadratic shape | 2.8464 | ( 0.3628 ) | 7.8447 |


Behavioral rate parameters:

- about 4 opportunities for a behavioral change in the first period
- about 4 opportunities for a behavioral change in the second period

## Parameter interpretation: a baseline model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Network Dynamics* | | | |
| constant friendship rate (period 1) | 8.6287 | ( 0.6666 ) | |
| constant friendship rate (period 2) | 7.2489 | ( 0.5466 ) | |
| outdegree (density) | -2.4084 | ( 0.0407 ) | -59.1268 |
| reciprocity | 2.7024 | ( 0.0823 ) | 32.8337 |
| *Behavior Dynamics* | | | |
| rate smokebeh (period 1) | 3.8922 | ( 1.9689 ) | |
| rate smokebeh (period 2) | 4.4813 | ( 2.3679 ) | |
| behavior smokebeh linear shap | -3.5464 | ( 0.4394 ) | -8.0712 |
| behavior smokebeh quadratic shape | 2.8464 | ( 0.3628 ) | 7.8447 |

Behavioral objective function parameters:

attractiveness of different behavioral levels based on the current structure of the network and the behavior of the others



## Parameter interpretation: a baseline model

- Smoking behavior: coded with 1 for "no", 2 for "occasional", and 3 for "regular" smokers.

- The smoking covariate is centered: $\overline{z} = 1.377$ is the mean of the covariate

$$z_i - \overline{z} = \begin{cases} -0.377 & \textit{for no smokers} \\ 0.623 & \textit{for occasional smokers} \\ 1.623 & \textit{for regular smokers} \end{cases}$$

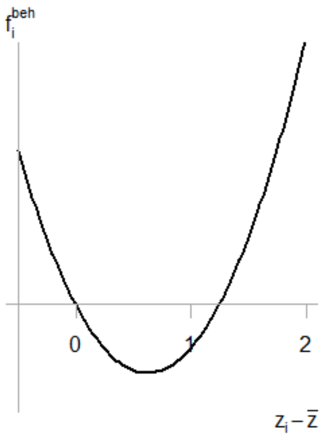
- The contribution to the behavioral objective function is

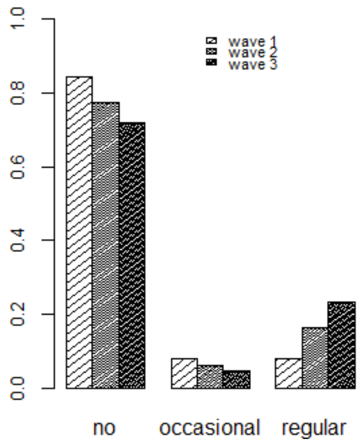
$$\gamma_{linear}(z_i - \overline{z}) + \gamma_{quadratic}(z_i - \overline{z})^2 =$$

$$= -3.5464(z_i - \overline{z}) + 2.8464(z_i - \overline{z})^2$$



## Parameter interpretation: a baseline model




U-shaped changes in the behavior are drawn to the extreme of the range



## A more complex model

The baseline model does not provide any information about selection and influence processes:

- the network dynamics are explained by the preference towards creating and reciprocating ties
- the behavior dynamics are described only by the distribution of the behavior in the population

If we want to distinguish selection from influence we should include in the objective functions specification:

- the effects that capture the dependence of social network dynamics on actor's characteristic
- the effects that capture the dependence of behavior dynamics on social network

## A more complex model

**Effects for the dependence of network dynamics on actor's characteristic**

- pupils prefer to establish friendship relations with others that are similar to themselves $\rightarrow$ covariate similarity

This effect must be controlled for the sender and receiver effects of the covariate.

- Covariate ego effect

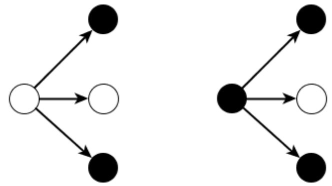
- Covariate alter effect



## A more complex model

**Effects for the dependence of behavior dynamics on network**

- pupils tend to adjust their smoking behavior according to the behaviors of their friends $\rightarrow$ average similarity effect



This effect must be controlled for the indegree and the outdegree effects

- Indegree effect

- Outdegree effect



## A more complex model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Network Dynamics* | | | |
| constant friendship rate (period 1) | 10.7166 | ( 1.4036 ) | |
| constant friendship rate (period 2) | 9.0005 | ( 0.7709 ) | |
| outdegree (density) | -2.8435 | ( 0.0572 ) | -49.6776 |
| reciprocity | 1.9683 | ( 0.0933 ) | 21.1077 |
| transitive triplets | 0.4447 | ( 0.0322 ) | 13.7964 |
| sex ego | 0.1612 | ( 0.1206 ) | 1.3368 |
| sex alter | -0.1476 | ( 0.1064 ) | -1.3871 |
| sex similarity | 0.9104 | ( 0.0882 ) | 10.3244 |
| smoke ego | 0.0665 | ( 0.0846 ) | 0.7857 |
| smoke alter | 0.1121 | ( 0.0761 ) | 1.4719 |
| smokebeh similarity | 0.5114 | ( 0.1735 ) | 2.9479 |

Network objective function parameters:

tendency towards reciprocity, transitivity and homophily with respect to gender



## A more complex model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Network Dynamics* | | | |
| constant friendship rate (period 1) | 10.7166 | ( 1.4036 ) | |
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| smoke alter | 0.1121 | ( 0.0761 ) | 1.4719 |
| smokebeh similarity | 0.5114 | ( 0.1735 ) | 2.9479 |

Network objective function parameters:

pupils selected others with similar smoking behavior as friends

$\rightarrow$ evidence for selection process

## A more complex model

The contribution to the network objective function is given by:

$$\beta_{ego}(z_i - \overline{z}) + \beta_{alter}(z_j - \overline{z}) + \beta_{same}\left(1 - \frac{|z_i - z_j|}{R_z} - sim_z\right) =$$

$$= 0.0665(z_i - 1.377) + 0.1121(z_j - 1.377) + 0.5114(1 - \frac{|z_i - z_j|}{R_z} - 0.7415)$$

| $z_i/z_j$ | no | occasional | regular |
|---|---|---|---|
| no | 0.0648 | -0.0787 | -0.2223 |
| occasional | -0.1243 | 0.2435 | 0.0999 |
| regular | -0.3135 | 0.0543 | 0.4221 |

- preference for similar alters
- this tendency is strongest for high values on smoking behavior



## A more complex model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Behavior Dynamics* | | | |
| rate smokebeh (period 1) | 3.9041 | (1.7402 ) | |
| rate smokebeh (period 2) | 3.8059 | ( 1.4323 ) | |
| behavior smokebeh linear shape | -3.3573 | ( 0.5678 ) | -5.9129 |
| behavior smokebeh quadratic shape | 2.8406 | ( 0.4125 ) | 6.8864 |
| behavior smokebeh indegree | 0.1711 | ( 0.1812 ) | 0.9444 |
| behavior smokebeh outdegree | 0.0128 | ( 0.1926 ) | 0.0662 |
| behavior smokebeh average similarity | 3.4361 | ( 1.4170 ) | 2.4250 |

Behavioral objective function parameters:

U-shaped distribution of the smoking behavior



## A more complex model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Behavior Dynamics* | | | |
| rate smokebeh (period 1) | 3.9041 | (1.7402 ) | |
| rate smokebeh (period 2) | 3.8059 | ( 1.4323 ) | |
| behavior smokebeh linear shape | -3.3573 | ( 0.5678 ) | -5.9129 |
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| behavior smokebeh indegree | 0.1711 | ( 0.1812 ) | 0.9444 |
| behavior smokebeh outdegree | 0.0128 | ( 0.1926 ) | 0.0662 |
| behavior smokebeh average similarity | 3.4361 | ( 1.4170 ) | 2.4250 |

Behavioral objective function parameters:

indegree and outdegree effects are not significant



## A more complex model

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Behavior Dynamics* | | | |
| rate smokebeh (period 1) | 3.9041 | (1.7402 ) | |
| rate smokebeh (period 2) | 3.8059 | ( 1.4323 ) | |
| behavior smokebeh linear shape | -3.3573 | ( 0.5678 ) | -5.9129 |
| behavior smokebeh quadratic shape | 2.8406 | ( 0.4125 ) | 6.8864 |
| behavior smokebeh indegree | 0.1711 | ( 0.1812 ) | 0.9444 |
| behavior smokebeh outdegree | 0.0128 | ( 0.1926 ) | 0.0662 |
| behavior smokebeh average similarity | 3.4361 | ( 1.4170 ) | 2.4250 |

Behavioral objective function parameters:

pupils are influenced by the smoking behavior of the others

$\rightarrow$ evidence for influence process

## A more complex model

The contribution to the behavioral objective function is given by:

$$\gamma_{linear}(z_i - \overline{z}) + \gamma_{quadratic}(z_i - \overline{z})^2 + \gamma_{avsim}\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - sim_z) =$$

$$= -3.3573(z_i - \overline{z}) + 2.8406(z_i - \overline{z})^2 + 3.4361\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - 0.7415)$$

where $sim_z(ij) = 1 - \frac{|z_i - z_j|}{R_Z} = 1$

### Example

a) $i$ adjusts his behavior to "no-smoker" when all of his friends are no-smokers

$$sim_z(ij) = 1 - \frac{|1-1|}{2} = 1$$

$$-3.3573(1 - 1.377) + 2.8406(1 - 1.377)^2 + 3.4361(1 - 0.7415) = 2.56$$



## A more complex model

The contribution to the behavioral objective function is given by:

$$\gamma_{linear}(z_i - \overline{z}) + \gamma_{quadratic}(z_i - \overline{z})^2 + \gamma_{avsim}\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - sim_z) =$$

$$= -3.3573(z_i - \overline{z}) + 2.8406(z_i - \overline{z})^2 + 3.4361\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - 0.7415)$$

where $sim_z(ij) = 1 - \frac{|z_i - z_j|}{R_Z} = 1$

### Example

b) $i$ adjusts his behavior to "no-smoker" when all of his friends are occasional smokers

$$sim_z(ij) = 1 - \frac{|1-2|}{2} = 0.5$$

$$-3.3573(1 - 1.377) + 2.8406(1 - 1.377)^2 + 3.4361(0.5 - 0.7415) = 0.84$$



## A more complex model

The contribution to the behavioral objective function is given by:

$$\gamma_{linear}(z_i - \overline{z}) + \gamma_{quadratic}(z_i - \overline{z})^2 + \gamma_{avsim}\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - sim_z) =$$

$$= -3.3573(z_i - \overline{z}) + 2.8406(z_i - \overline{z})^2 + 3.4361\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - 0.7415)$$

where $sim_z(ij) = 1 - \frac{|z_i - z_j|}{R_Z} = 1$

### Example

b) $i$ adjusts his behavior to "no-smoker" when all of his friends are regular smokers

$$sim_z(ij) = 1 - \frac{|1-3|}{2} = 0$$

$$-3.3573(1 - 1.377) + 2.8406(1 - 1.377)^2 + 3.4361(0 - 0.7415) = -0.88$$



## A more complex model

The contribution to the behavioral objective function is given by:

$$\gamma_{linear}(z_i - \overline{z}) + \gamma_{quadratic}(z_i - \overline{z})^2 + \gamma_{avsim}\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - sim_z) =$$

$$= -3.3573_{linear}(z_i - \overline{z}) + 2.8406_{quadratic}(z_i - \overline{z})^2 + 3.4361\frac{1}{x_{i+}}\sum_{j=1}^{n} x_{ij}(sim_z(ij) - 0.7415)$$

| $z_j$ / $z_i$ | no | occasional | regular |
|---|---|---|---|
| no | 2.56 | -1.82 | -0.51 |
| occasional | 0.84 | -0.10 | 1.20 |
| regular | -0.88 | -1.82 | 2.92 |

- the focal actor prefers to have the same behavior as all these friends (except for the occasional smokers)

- friends do not smoke at all: the preference toward imitating their behavior is less strong

## Simulating the co-evolution of networks and behavior

**Algorithm 5:** Co-evolution of networks and behavior

**Input**: $x(t_0)$, $z(t_0)$, $\lambda^{net}$, $\lambda^{beh}$, $\beta$, $\gamma$, $n$
**Output**: $x^{sim}(t_1)$, $z^{sim}(t_1)$
$t \leftarrow 0$; $x \leftarrow x(t_0)$; $z \leftarrow z(t_0)$
**while** *condition=TRUE* **do**
  $T^{net} \sim Exp(\lambda^{net})$, $T^{beh} \sim Exp(\lambda^{beh})$
  **if** $\min\{T^{net}, T^{beh}\} = T^{net}$ **then**
    $i \sim Uniform(1, \ldots, n)$, $j \sim p_{ij}$
    **if** $i \neq j$ **then**
      $x \leftarrow x(i \rightsquigarrow j)$
    **else**
      $x \leftarrow x$
    $t \leftarrow t + T^{net}$
  **else**
    $i \sim Uniform(1, \ldots, n)$, $l' \sim p_{ll'}$
    **if** $l \neq l'$ **then**
      $z \leftarrow z(l \rightsquigarrow l')$
    **else**
      $z \leftarrow z$
    $t \leftarrow t + T^{beh}$
$x^{sim}(t_1) \leftarrow x$
$z^{sim}(t_1) \leftarrow z$
**return** $x^{sim}(t_1)$, $z^{sim}(t_1)$



## Simulating the co-evolution of networks and behavior

1. *Unconditional* simulation:

   simulation carries on until a predetermined time length has elapsed (usually until $t = 1$).

2. *Conditional* simulation on the observed number of changes:
   - simulation runs on until

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left| X_{ij}^{obs}(t_1) - X_{ij}(t_0) \right| = \sum_{i,j=1}^{n} \left| X_{ij}^{sim}(t_1) - X_{ij}(t_0) \right|$$

   - simulation runs on until

$$\sum_{i=1}^{n} \left| z_i^{obs}(t_1) - z_i(t_0) \right| = \sum_{i=1}^{n} \left| z_i^{sim}(t_1) - z_i(t_0) \right|$$



## The parameter estimation (MoM)

Aim: estimate the parameter $\theta$ for the co-evolution model

- $M-1$ rate parameters for the network rate function

$$\lambda_1^{net}, \ \ldots, \ \lambda_{M-1}^{net}$$

- $M-1$ rate parameters for the behavior rate function

$$\lambda_1^{beh}, \ \ldots, \ \lambda_{M-1}^{beh}$$

- $K$ parameters for the network objective function

$$\sum_{k=1}^{K} \beta_k s_{ik}^{net}(x, z, v)$$

- $W$ parameters for the behavior objective function

$$\sum_{w=1}^{W} \gamma_w s_{iw}^{beh}(x, z(l \rightsquigarrow l'), v)$$



## The parameter estimation (MoM)

Aim: estimate the $2(M-1) + K + W$-dimensional parameter $\theta$ using the MoM

In practice:

1. find $2(M-1) + K + W$ statistics
2. set the theoretical expected value of each statistic equal to its sample counterpart
3. solve the resulting system of equations

$$E_\theta[S - s] = 0$$

   with respect to $\theta$

## The parameter estimation (MoM)

**Statistics:**

- Network rate parameters for the period $m$

$$s^{net}_{\lambda_m}(X(t_m), X(t_{m-1}) | X(t_{m-1}) = x(t_{m-1})) = \sum_{i,j=1}^{n} \left| X_{ij}(t_m) - X_{ij}(t_{m-1}) \right|$$

- Behavior rate parameters for the period $m$

$$s^{beh}_{\lambda_m}(Z(t_m), Z(t_{m-1}) | Z(t_{m-1}) = z(t_{m-1})) = \sum_{i=1}^{n} |Z_i(t_m) - Z_i(t_{m-1})|$$



## The parameter estimation (MoM)

**Statistics:**

- Network objective function effects

$$\sum_{m=1}^{M-1} s^{net}_{mk}((X,Z,V)(t_m) | (x,z,v)(t_{m-1})) = \sum_{m=1}^{M-1} s^{net}_{mk}((X,Z,V)(t_m), (X,Z,V)(t_{m-1}))$$

- Behavior objective function effects

$$\sum_{m=1}^{M-1} s^{beh}_{mw}((X,Z,V)(t_m) | (x,z,v)(t_{m-1})) = \sum_{m=1}^{M-1} s^{beh}_{mw}((X,Z,V)(t_m), (X,Z,V)(t_{m-1}))$$



## The parameter estimation (MoM)

Consequently the MoM estimator for $\theta$ is provided by the solution of the system of equations:

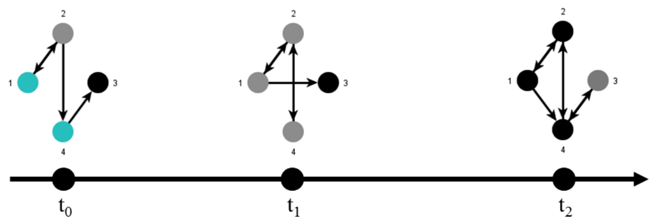
$$\begin{cases} E_\theta\left[s_{\lambda_m}(X(t_m), X(t_{m-1}) | X(t_{m-1}) = x(t_{m-1}))\right] = s_{\lambda_m}(x(t_m), x(t_{m-1})) \\[2ex] E_\theta\left[s_{\lambda_m}(Z(t_m), Z(t_{m-1}) | Z(t_{m-1}) = z(t_{m-1}))\right] = s_{\lambda_m}(z(t_m), z(t_{m-1})) \\[2ex] E_\theta\left[\sum_{m=1}^{M-1} s^{net}_{mk}((X,Z,V)(t_m) | (x,z,v)(t_{m-1}))\right] = \sum_{m=1}^{M-1} s^{net}_{mk}((x,z,v)(t_m), (x,z,v)(t_{m-1})) \\[2ex] E_\theta\left[\sum_{m=1}^{M-1} s^{beh}_{mw}((X,Z,V)(t_m) | (x,z,v)(t_{m-1}))\right] = \sum_{m=1}^{M-1} s^{beh}_{mw}((x,z,v)(t_m), (x,z,v)(t_{m-1})) \end{cases}$$



## The parameter estimation (MoM)

### Example

Let us assume to have observed a network at $M = 3$ time points




$t_0$:

| | 1 | 2 | 3 | 4 | beh |
|---|---|---|---|---|---|
| 1 | - | 1 | 0 | 0 | 0 |
| 2 | 1 | - | 0 | 1 | 1 |
| 3 | 0 | 0 | - | 0 | 2 |
| 4 | 0 | 0 | 1 | - | 0 |

$t_1$:

| | 1 | 2 | 3 | 4 | beh |
|---|---|---|---|---|---|
| 1 | - | 1 | 1 | 0 | 1 |
| 2 | 1 | - | 0 | 1 | 1 |
| 3 | 0 | 0 | - | 0 | 2 |
| 4 | 0 | 1 | 0 | - | 1 |

$t_2$:

| | 1 | 2 | 3 | 4 | beh |
|---|---|---|---|---|---|
| 1 | - | 1 | 0 | 1 | 2 |
| 2 | 1 | - | 0 | 1 | 2 |
| 3 | 0 | 0 | - | 1 | 1 |
| 4 | 0 | 1 | 1 | - | 2 |

We want to model the network evolution according to the outdegree, the reciprocity, the linear shape and the quadratic shape effects

$$\theta = (\lambda^{net}_1, \lambda^{net}_2, \lambda^{beh}_1, \lambda^{beh}_2, \beta_{out}, \beta_{rec}, \gamma_{linear}, \gamma_{quadratic})$$

## The parameter estimation (MoM)

### Example

Statistics for the network evolution:

$$s_{\lambda_1^{net}}(X(t_1), X(t_0)|X(t_0) = x(t_0)) = \sum_{i,j=1}^{4} \left|X_{ij}(t_1) - X_{ij}(t_0)\right|$$

$$s_{\lambda_2^{net}}(X(t_2), X(t_1)|X(t_1) = x(t_1)) = \sum_{i,j=1}^{4} \left|X_{ij}(t_2) - X_{ij}(t_1)\right|$$

$$\sum_{m=1}^{M-1} s_{out}\left(X(t_m)|X(t_{m-1}) = x(t_{m-1})\right) = \sum_{m=1}^{2} \sum_{i,j=1}^{4} X_{ij}(t_m)$$

$$\sum_{m=1}^{M-1} s_{rec}\left(X(t_m)|X(t_{m-1}) = x(t_{m-1})\right) = \sum_{m=1}^{2} \sum_{i,j=1}^{4} X_{ij}(t_m)X_{ji}(t_m)$$



## The parameter estimation (MoM)

### Example

Statistics for the behavior evolution:

$$s_{\lambda_1^{beh}}(Z(t_1), Z(t_0)|Z(t_0) = z(t_0)) = \sum_{i=1}^{4} |Z_i(t_1) - Z_i(t_0)|$$

$$s_{\lambda_2^{beh}}(Z(t_2), Z(t_1)|Z(t_1) = z(t_1)) = \sum_{i=1}^{4} |Z_i(t_2) - Z_i(t_1)|$$

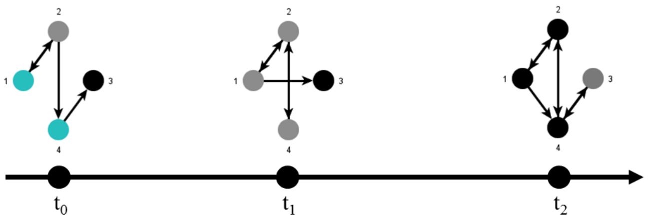
$$\sum_{m=1}^{M-1} s_{linear}\left(Z(t_m)|Z(t_{m-1}) = z(t_{m-1})\right) = \sum_{m=1}^{2} \sum_{i=1}^{4} z_i(t_m)$$

$$\sum_{m=1}^{M-1} s_{quadratic}\left(Z(t_m)|Z(t_{m-1}) = z(t_{m-1})\right) = \sum_{m=1}^{2} \sum_{i=1}^{4} z_i^2(t_m)$$



## The parameter estimation (MoM)

### Example




$t_0$:

| | 1 | 2 | 3 | 4 | beh |
|---|---|---|---|---|---|
| 1 | - | 1 | 0 | 0 | 0 |
| 2 | 1 | - | 0 | 1 | 1 |
| 3 | 0 | 0 | - | 0 | 2 |
| 4 | 0 | 0 | 1 | - | 0 |

$t_1$:

| | 1 | 2 | 3 | 4 | beh |
|---|---|---|---|---|---|
| 1 | - | 1 | 1 | 0 | 1 |
| 2 | 1 | - | 0 | 1 | 1 |
| 3 | 0 | 0 | - | 0 | 2 |
| 4 | 0 | 1 | 0 | - | 1 |

$t_2$:

| | 1 | 2 | 3 | 4 | beh |
|---|---|---|---|---|---|
| 1 | - | 1 | 0 | 1 | 2 |
| 2 | 1 | - | 0 | 1 | 2 |
| 3 | 0 | 0 | - | 1 | 1 |
| 4 | 0 | 1 | 1 | - | 2 |

$s_{\lambda_1^{net}} = 3$ $\qquad$ $s_{\lambda_2^{net}} = 4$

$s_{\lambda_1^{beh}} = 2$ $\qquad$ $s_{\lambda_2^{beh}} = 4$

$s_{out} = 5 + 7 = 12$ $\qquad$ $s_{rec} = 4 + 6 = 10$

$s_{linear} = 5 + 7 = 12$ $\qquad$ $s_{quadratic} = 7 + 13 = 20$



## The parameter estimation (MoM)

### Example

We look for the value of $\theta$ that satisfies the system:

$$\begin{cases} E_\theta\left[S_{\lambda_1^{net}}\right] = 3 \\ E_\theta\left[S_{\lambda_2^{net}}\right] = 4 \\ E_\theta\left[S_{\lambda_1^{beh}}\right] = 2 \\ E_\theta\left[S_{\lambda_2^{beh}}\right] = 4 \\ E_\theta[S_{out}] = 12 \\ E_\theta[S_{rec}] = 10 \\ E_\theta[S_{linear}] = 12 \\ E_\theta[S_{quadratic}] = 20 \end{cases}$$

## The parameter estimation (MoM)

In a more compact notation, we look for the value of $\theta$ that satisfies the system:

$$E_\theta[S - s] = 0$$

but we know that we cannot solve it analytically.
We can use the Robbins-Monro algorithm:

*Phase 1*: provide the initial value for $\theta$ and for $D$

*Phase 2*: updating the value of $\theta$ via the RM step:

$$\widehat{\theta}_{i+1} = \widehat{\theta}_i - a_i D^{-1} \left(E_{\theta_i}[S] - s\right)$$

*Phase 3*: estimate the s.e. of the estimate



## Outline





## Creating and deleting ties

Terminating a tie is not just the opposite of creating a tie

### Example

- the loss in terminating a tie is greater than the reward in creating one
- transitivity plays an important role especially in creating ties

This is modeled by adding to the objective function one of the two components:

1. the *creation function*
2. the *endowment function*



## The creation function

Models the gain in satisfaction incurred when a network tie is created:

$$c_i(\delta, x) = \sum_k \delta_k s_{ik}(x)$$

where

- $\delta_k$ are statistical parameters
- $s_{ik}(x)$ are the effects

The utility function for an actor $i$ when he creates a new tie is provided by:

$$u_i(x) = f_i(\beta, x) + c_i(\delta, x) + \epsilon_i(t, x, j)$$

The creation function is zero for the dissolution of ties

## The endowment function

Models the loss in satisfaction incurred when a network tie is deleted

$$e_i(\eta, x) = \sum_k \eta_k s_{ik}(x)$$

where

- $\eta_k$ are statistical parameters
- $s_{ik}(x)$ are the effects

The utility function for an actor $i$ when he deletes a tie is provided by:

$$u_i(x) = f_i(\beta, x) + e_i(\eta, x) + \epsilon_i(t, x, j)$$

The endowment function is zero for the creation of ties



## Creating and deleting ties - Remarks

- creation and deletion functions must not be included when ties mainly are created or terminated
- it could also happen that increasing a behavior is not the same as decreasing a behavior. Thus, there are also:
  1. the *creation behavior function*
  2. the *endowment behavior function*

  but their usage is still under investigation



## Creating and deleting ties

### Example

Example data: excerpt from the “Teenage Friends and Lifestyle Study” data set

We estimate the SAOM for investing the evolution of friendship networks according to:

- outdegree
- reciprocity
- transitivity
- reciprocity for the endowment function

```
myeff <- includeEffects(myeff,transTrip)
myeff <- includeEffects(myeff,recip,type="endow")
myeff
mymodel <- sienaModelCreate(useStdInits = FALSE, projname = 'tfls')
model1 <- siena07(mymodel, data = mydata, effects=myeff,useCluster=TRUE,
nbrNodes=2, initC=TRUE,clusterString=rep("localhost", 2))
```



## Creating and deleting ties

### Example

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 6.70 | 0.73 | |
| Rate parameter period 2 | 5.81 | 0.58 | |
| *Other parameters:* | | | |
| outdegree | -2.58 | 0.05 | -51.62 |
| reciprocity | 3.23 | 0.29 | 11.15 |
| reciprocity (endow) | -2.23 | 0.58 | -3.85 |
| transitive triplets | 0.44 | 0.03 | 14.55 |

The utility function for an actor $i$ when he deletes a tie is provided by:

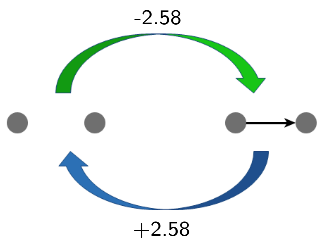
$$\begin{aligned} u_i(x) &= f_i(\beta, x) + e_i(\eta, x) + \epsilon_i(t, x, j) = \\ &= \beta_{out} s_{i\_out}(x) + \beta_{rec} s_{i\_rec}(x) + \beta_{trans} s_{i\_trans}(x) + \eta_{rec} s_{i\_rec}(x) \\ &= -2.58 s_{i\_out}(x) + 3.23 s_{i\_rec}(x) + 0.44 s_{i\_trans}(x) - 2.23 s_{i\_rec}(x) \end{aligned}$$

## Creating and deleting ties

### Example

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 8.44 | 0.73 | |
| Rate parameter period 2 | 7.09 | 0.58 | |
| *Other parameters:* | | | |
| outdegree | -2.58 | 0.05 | -51.62 |
| reciprocity | 3.23 | 0.29 | 11.15 |
| reciprocity (endow) | -2.23 | 0.58 | -3.85 |
| transitive triplets | 0.44 | 0.03 | 14.55 |

Ties formation/deletion






## Creating and deleting ties

### Example

| | Estimates | s.e. | t-score |
|---|---|---|---|
| *Rate parameters:* | | | |
| Rate parameter period 1 | 8.44 | 0.73 | |
| Rate parameter period 2 | 7.09 | 0.58 | |
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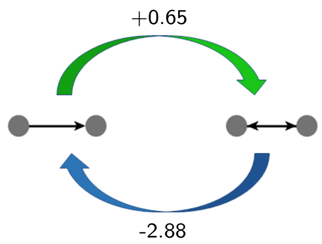
| transitive triplets | 0.44 | 0.03 | 14.55 |

Reciprocation/ending reciprocation

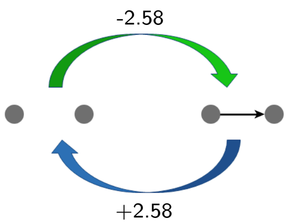




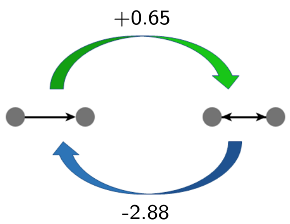

## Creating and deleting ties

### Example

Ties formation/deletion




Reciprocation/ending reciprocation




Conclusions:

1. formation of reciprocal ties is more rewarding than the formation of a non-reciprocal tie
2. dissolution of reciprocal ties is more costly than the dissolution of a non-reciprocal tie and the creation of a reciprocal tie



## Non-directed relations

For directed relation we assumed that:

1. an actor gets the opportunity to make a change
2. he decided for the change that assures him the highest payoff

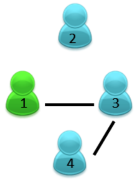
Are this assumptions still reliable when we consider undirected relations such as: collaboration, trade, strategic alliance?

- Yes, if one actor (*dictator*) can impose a decision about a tie to another
- No, if there is coordination or negotiation about a tie

## Non-directed relations

1. Dictatorial choice: $i$ chooses his action and imposes his decision to $j$

Actor 1 gets the opportunity to change



| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 0 | 1 | 0 |
| 2 | 0 | - | 0 | 0 |
| 3 | 1 | 0 | - | 1 |
| 4 | 0 | 0 | 1 | - |

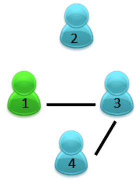
x = current state of the network



## Non-directed relations

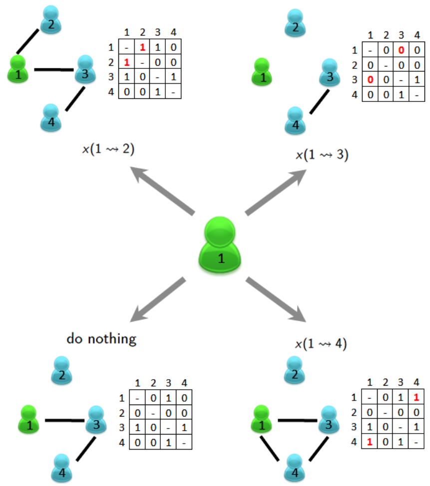
1. Dictatorial choice: $i$ chooses his action and imposes his decision to $j$

Actor 1 evaluates the alternatives and the corresponding objective functions



| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 0 | 1 | 0 |
| 2 | 0 | - | 0 | 0 |
| 3 | 1 | 0 | - | 1 |
| 4 | 0 | 0 | 1 | - |

x = current state of the network

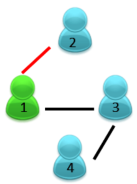





## Non-directed relations

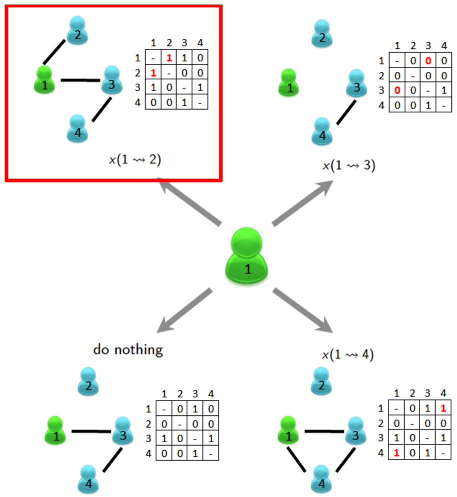
1. Dictatorial choice: $i$ chooses his action and imposes his decision to $j$

E.g. actor 1 imposes his choice to actor 1



| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 0 | 1 | 0 |
| 2 | 0 | - | 0 | 0 |
| 3 | 1 | 0 | - | 1 |
| 4 | 0 | 0 | 1 | - |

x = current state of the network

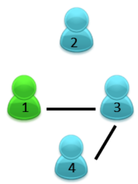





## Non-directed relations

2. Mutual agreement: both actors must agree

Actor 1 gets the opportunity to change



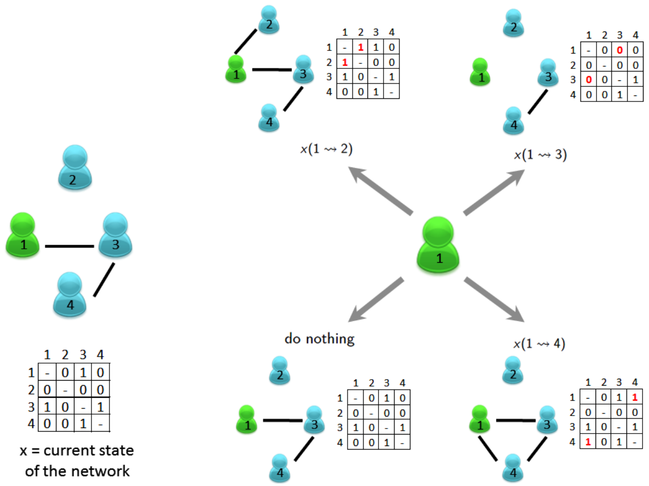
| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | 0 | 1 | 0 |
| 2 | 0 | - | 0 | 0 |
| 3 | 1 | 0 | - | 1 |
| 4 | 0 | 0 | 1 | - |

x = current state of the network

## Non-directed relations

2. Mutual agreement: both actors must agree

Actor 1 evaluates the alternatives and the corresponding objective functions

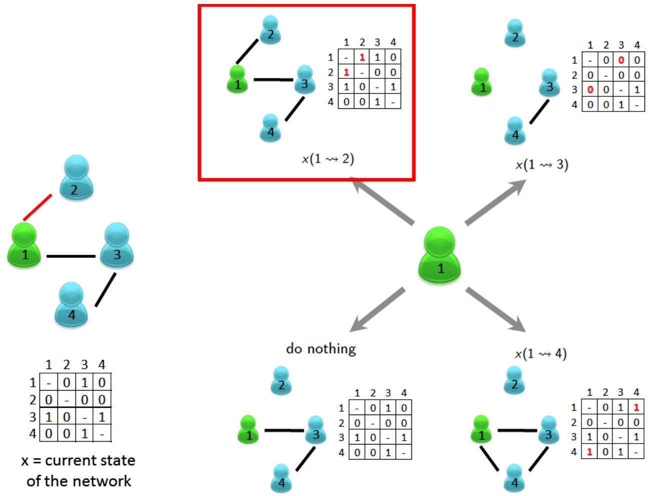





## Non-directed relations

2. Mutual agreement: both actors must agree

Actor 1 suggests to modify the tie towards actor 2

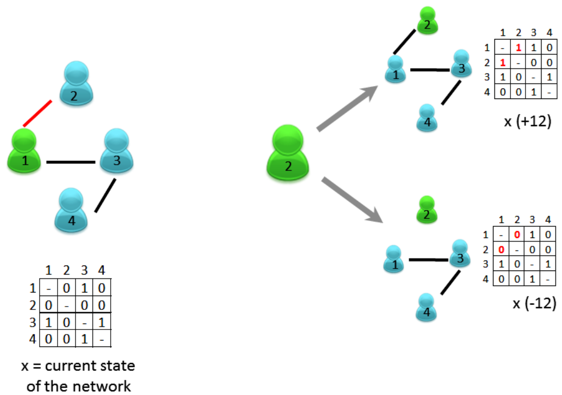





## Non-directed relations

2. Mutual agreement: both actors must agree

Actor 2 evaluates the proposal of actor 1




and accepts it with probability

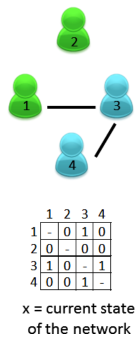
$$P(2\ accepts\ tie\ proposal) = \frac{exp(f_2(x^{+12}))}{exp(f_2(x^{+12})) + exp(f_2(x^{-12}))}$$



## Non-directed relations - Tie-based approach

A couple $(i, j)$ of actors is selected with rate $\lambda_{ij}$ and gets the opportunity to revise the tie among them

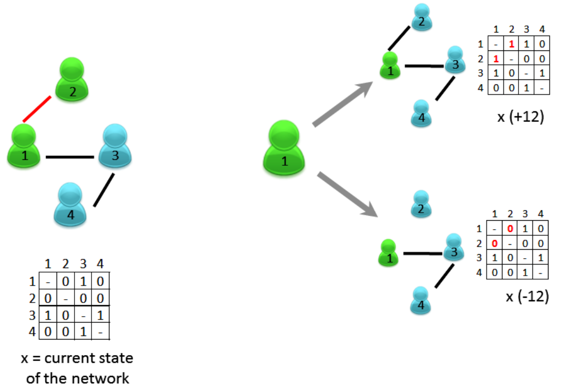
1. Dictatorial choice: one actor can impose the decision (e.g. actor 1)



## Non-directed relations - Tie-based approach

A couple $(i, j)$ of actors is selected with rate $\lambda_{ij}$ and gets the opportunity to revise the tie among them

1. Dictatorial choice: one actor can impose the decision (e.g. actor 1)




Actor 1 chooses his action with probability

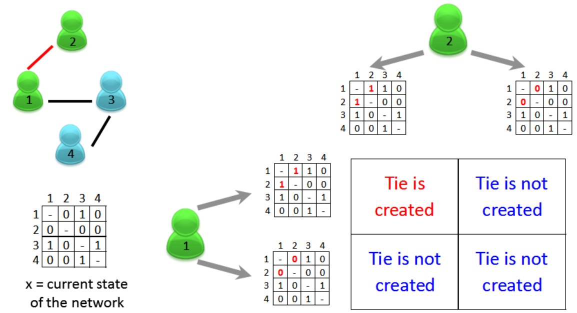
$$P(1 \ imposes \ a \ tie \ on \ 2) = \frac{exp(f_1(x^{+12}))}{exp(f_1(x^{+12})) + exp(f_1(x^{-12}))}$$



## Non-directed relations - Tie-based approach

A couple $(i, j)$ of actors is selected with rate $\lambda_{ij}$ and gets the opportunity to revise the tie among them

2. Mutual agreement: both actors propose a tie




Actor 1 and 2 created a tie with probability

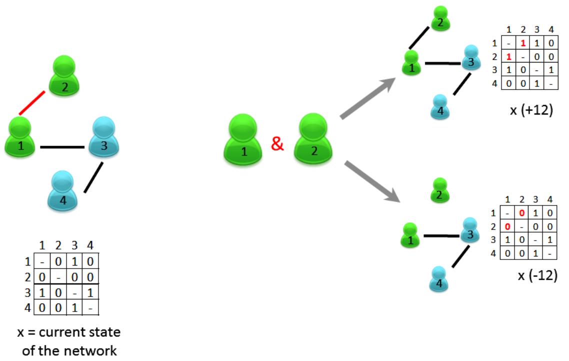
$$P(+12) = \frac{exp(f_1(x^{+12}))}{exp(f_1(x^{+12})) + exp(f_1(x^{-12}))} \frac{exp(f_2(x^{+12})}{exp(f_2(x^{+12})) + exp(f_2(x^{-12}))}$$



## Non-directed relations - Tie-based approach

A couple $(i, j)$ of actors is selected with rate $\lambda_{ij}$ and gets the opportunity to revise the tie among them

3. Compensatory: the decision is made on the combined interest




Actor 1 and 2 choose their action with probability

$$P(+12) = \frac{exp(f_1(x^{+12}) + f_2(x^{+12}))}{exp(f_1(x^{+12}) + f_2(x^{+12})) + exp(f_1(x^{-12}) + f_2(x^{-12}))}$$



## And others...

- Improving the estimation procedures (MLE)
- New estimation procedures (bayesian estimation)
- Goodness of fit of the model
- Model selection
- Time-heterogeneity tests
- Missing data
- Analysis of multiple relations
- ...