University of Konstanz
Department of Computer & Information Science
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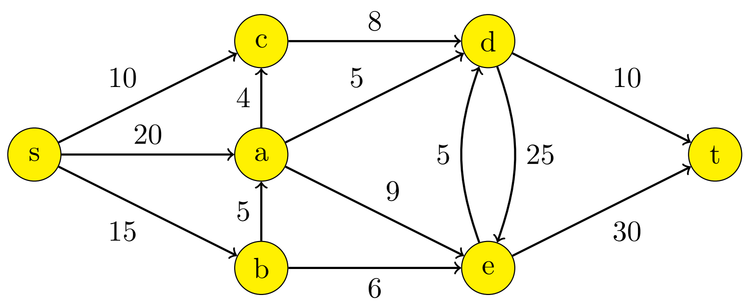
Design and Analysis of Algorithms
Winter 2018/2019

Assignment 7

Post Date: 03 Dec 2018 **Due Date:** 10 Dec 2018, 11:30 am
You are permitted and encouraged to work in groups of two.

Problem 1: Computing a Maximum Flow — 6 Points

Consider the following flow network where the numbers at the edges denote their capacity:



Find a maximum s-t-flow using the algorithm of Goldberg and Tarjan. Indicate in each step: the selected active vertex, the selected basic operation with the involved vertices, and depending on the operation, the value Δ for the corresponding edge and the new flow, or the h-labels for the vertices where the labels were changed.

Problem 2: Santa Claus Problem — 8 Points

In order to have enough presents for all children, Santa Claus prepares presents all year round. At the beginning of December he obtains wish lists from all the children. Now Santa Claus tries to assign each child a present such that as many children as possible obtain a present they are happy with.

Santa Claus' IT department suggests to use the following max-flow network $(D, c \equiv 1, s, t)$. The vertex set is $V = \{s,t\} \cup C \cup P$ where C is the set of children and P is the set of presents. There are the following edges with unit capacity: $(s, c), c \in C$, (p, t), $p \in P$, and (c, p), if child c has present p on its wish list.

(a) Show that an optimum solution of Santa Claus' problem can indeed be constructed from a maximum flow in this flow network. How fast can an optimum assignment of presents be found?

(b) Show that D has a minimum cut $(S, V \setminus S)$ with the property that there is no edge between $C \cap S$ and $P \setminus S$.

[please turn over]

(c) For a child c let $W(c)$ be the set of presents on its wish list. For a set C' of children let $W(C') = \bigcup_{c \in C'} W(c)$. Use the max-flow min-cut theorem to show that Santa Claus can assign the presents such that all children receive a present they like if and only if there is no subset C' of children with $|W(C')| < |C'|$.

Problem 3: FIFO Vertex Selection Rule **6 Points**

Show that the run time of the algorithm of Goldberg & Tarjan applied to a flow network with n vertices is in $\mathcal{O}(n^3)$ if it is implemented by using the FIFO vertex selection rule as follows: First all vertices that are activated during the initialization are appended to a queue. While the queue is not empty the algorithm removes the first vertex v from the queue, performs PUSH-operations from v and appends newly active vertices to the queue. The algorithm examines v until it is not active any more or a RELABEL-operation is performed. In the latter case v is appended again to the queue.

Hint: Partition the vertex examinations into phases. The first phase consists of examinations of vertices that become active during the initialization. The $(i+1)$st phase consists of examinations of vertices that were appended to the queue during the ith phase.

(a) Show that there are $\mathcal{O}(n^2)$ phases in which a RELABEL-operation is executed.

(b) Use the potential

$$\Phi = \max_{v \text{ active}} h(v).$$

to show that there are $\mathcal{O}(n^2)$ phases in which no RELABEL-operation is executed.

(c) Estimate the number of non-saturating PUSH-operations per phase.