University of Konstanz
Department of Computer & Information Science
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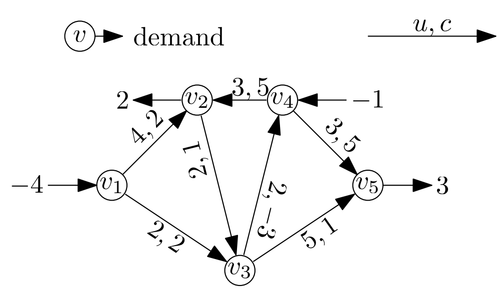
Design and Analysis of Algorithms
Winter 2018/2019

Assignment 8

Post Date: 10 Dec 2018 **Due Date:** 17 Dec 2018, 11:30
You are permitted and encouraged to work in groups of two.

Problem 1: Successive Shortest-Paths I **8 Points**

Apply the Successive Shortest-Path algorithm to the min-cost flow problem below. Indicate in each step the flow, the deficiencies, and the potentials.



Problem 2: Successive Shortest-Paths II **6 Points**

Show that if the Successive Shortest-Path algorithm returns “no feasible flow” then there is no feasible flow.

Hint: Assume that the algorithm returns “no feasible flow” after having computed a pseudo-flow f and considering a vertex s' with negative deficiency. Let $\mathcal{N}_f$ be the residual network with respect to f and let $\mathcal{N}^{st}$ be the corresponding max-flow network of $\mathcal{N}_f$. Consider the cut of $\mathcal{N}^{st}$ induced by

$$\{s\} \cup \{w \in V \cup \{s,t\};\ \text{ex. directed } s' - w\text{-path in } \mathcal{N}^{st}\}.$$

[please turn over]

Problem 3: All-Pairs Shortest Paths **6 Points**

On a directed graph $G = (V, E)$ with n vertices, m edges and edge weights $\omega : E \to \mathbb{R}$, the Single-Source Shortest Paths (SSSP) problem asks to find shortest paths from a single source vertex to all other vertices. If (G, ω) does not have any negative-weight edges, the algorithm of Dijkstra solves the SSSP problem on (G, ω) in $\mathcal{O}(m + n \log n)$ time. On graphs with negative-weight edges the SSSP problem is still well defined if there are no negative-weight cycles. In this case, the algorithm of Bellman and Ford computes the shortest paths from a single source vertex to all other vertices on such a graph (G, ω) in $\mathcal{O}(nm)$ time.

Now you need an algorithm that finds shortest paths between all pairs of vertices in a directed graph. This problem is known as the All-Pairs Shortest Paths (APSP) problem.

(a) Show how to solve the APSP problem on graphs with no negative-weight edges in $\mathcal{O}(nm + n^2 \log n)$ time.

(b) Assume now that you might have negative-weight edges, but no negative-weight cycles. Find a way to compute potentials $\pi : V \to \mathbb{R}$ on (G, ω) in $\mathcal{O}(nm)$ time such that the reduced weights $\omega_\pi(e) = \pi(v) + \omega(e) - \pi(w)$ are non-negative for any edge $e = (v, w) \in E$.

(c) Show how the APSP problem can be solved in $\mathcal{O}(nm + n^2 \log n)$ time on directed graphs with negative-weight edges, but no negative-weight cycles.

Note: You may use the algorithms of Dijkstra and of Bellman and Ford as building blocks. It is not necessary to understand or argue about the internals of these SSSP algorithms to solve the exercise.