

## Assignment 10

**Post Date:** 07 Jan 2019    **Due Date:** 14 Jan 2019, 11:30  
You are permitted and encouraged to work in groups of two.

### Problem 1: Truck Loading

4 Points

Suppose a ship arrives with  $n$  containers of positive weight  $w_1, \dots, w_n$ . You want to load the containers into trucks, each of which can hold cargo of weight at most  $K \geq \max_i w_i$ .

You apply the following scheme: Start with an empty truck and fill it with containers 1, 2, ... until the next container does not fit into the truck. Then send this truck off and continue the process by filling the next truck. Iterate this scheme until all containers were loaded into trucks.

Show that the number of trucks used by this scheme is at most twice the minimum number of trucks possible.

### Problem 2: Central Points

6 Points

In the  $\ell$ -central points problem, you are given a set  $P$  of  $n$  points in the plane with Euclidean distances  $d$  and a positive integer  $\ell$ . We call the points in  $P$  *sites*. You want to build at most  $\ell$  centers such that the maximum distance of any site to its nearest center is minimized. I.e. the feasible solutions are the sets  $C$  of at most  $\ell$  points in the plane, the cost of  $C$  is its *covering radius*  $r(C) = \max_{p \in P} \min_{c \in C} d(p, c)$  and you want to find a feasible solution with minimum cost.

- (a) Consider the following method: Start with the empty set  $C$ . Keep adding points to  $C$  reducing the covering radius each time by as much as possible. I.e., iteratively find a point  $c^*$  in the plane (not necessarily from  $P$ ) such that

$$r(C \cup \{c^*\}) = \min_{c' \in \mathbb{R}^2} r(C \cup \{c'\})$$

and add  $c^*$  to  $C$  until the size of  $C$  is  $\ell$ .

Does this yield a  $k$ -approximation for the  $\ell$ -central points problem for some constant  $k \geq 1$ ?

- (b) Assuming you knew the optimum covering radius  $r$ , consider the following algorithm. Start with  $C = \emptyset$  and  $P' = P$ . While  $P'$  is not empty, iteratively pick a site  $p \in P'$ , add it to  $C$ , and remove all points from  $P'$  that have distance at most  $2r$  from  $p$ .

Prove that this yields a 2-approximation for the  $\ell$ -central points problem.

[Please turn over]

- (c) Also consider the following algorithm. Select a site  $c \in P$  and start with  $C = \{c\}$ . Iteratively select a site  $p \in P$  that maximizes  $\min_{c \in C} d(p, c)$  and add  $p$  to  $C$  until the size of  $C$  is  $\ell$  or  $C = P$ .

Does this yield a  $k$ -approximation for the  $\ell$ -central points problem for some constant  $k \geq 1$ ?

### Problem 3: Approximating Knapsack

10 Points

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#### Algorithm 1: Approximation Algorithm for Knapsack

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**Input:** positive integers  $w_1, \dots, w_n, v_1, \dots, v_n, W$  such that  $v_1/w_1 \geq \dots \geq v_n/w_n$

**Output:** A feasible solution  $I \subseteq \{1, \dots, n\}$

$I \leftarrow \emptyset;$

**for**  $i = 1, \dots, n$  **do**

**if**  $\sum_{j \in I \cup \{i\}} w_j \leq W$  **then**  
 $I \leftarrow I \cup \{i\};$

**for**  $i = 1, \dots, n$  **do**

**if**  $w_i \leq W$  **and**  $v_i > v(I)$  **then**  
 $I \leftarrow \{i\};$

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- (a) Suppose you omit the first for loop of Algorithm 1. Show that such an algorithm is not a  $k$ -approximation algorithm for Knapsack for any  $k \in \mathbb{N}_+$ .
- (b) Suppose you omit the second for loop of Algorithm 1. Show that such an algorithm is not a  $k$ -approximation algorithm for Knapsack for any  $k \in \mathbb{N}_+$ .

In order to analyze the quality of the approximation produced by the algorithm, consider now the following continuous variant of the Knapsack problem:

**Instance:** weights  $(w_1, \dots, w_n) \in \mathbb{N}^n$ , values  $(v_1, \dots, v_n) \in \mathbb{N}^n$ , maximum weight  $W \in \mathbb{N}$ .

**feasible solutions:** vectors  $I = (x_1, \dots, x_n) \in [0, 1]^n$  with  $\sum_{i=1}^n x_i w_i \leq W$ .

**value of  $I$ :**  $v(I) = \sum_{i=1}^n x_i v_i$

**optimum solution:** a feasible solution with maximum value.

- (c) Suppose you are given an instance of the continuous Knapsack problem such that  $v_1/w_1 \geq \dots \geq v_n/w_n$ . Show that there exists an integer  $k \in \{1, \dots, n\}$  and an optimum solution  $(x_1, \dots, x_n)$  such that  $x_i = 1$  for  $1 \leq i < k$ ,  $0 \leq x_k \leq 1$ , and  $x_i = 0$  for  $n \geq i > k$ .
- (d) Show that the unmodified Algorithm 1 is a 2-approximation algorithm.  
 Hint: You may use (c), even if you did not prove it.
- (e) Is Algorithm 1 a  $k$ -approximation algorithm for some  $k < 2$ ? Prove your claim.